More on Levi-Civita connection and geodesics

Last time: (M,g) Riem mfd

$\Longrightarrow$ $\exists!$ orthogonal, torsion-free connection ∇ on TM.

In local coords, $\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) = \Gamma^k_{ij}\frac{\partial}{\partial x^k}$ $\qquad$ $(\Gamma^k_{ij}\, dx^i = A^k_j$ connection coeff$)$

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_k g_{ij}\right)$$

Prop The induced connection on a submanifold $M \subset \mathbb{R}^n$ (defined using orth. projection) coincides with the Levi-Civita connection. [Exercise]

Prop If $\varphi: M \to \tilde{M}$ is an isometry and $\nabla, \tilde{\nabla}$ are Levi-Civita then $\nabla = \varphi^*\tilde{\nabla}$.

Pf Just show $\varphi^*\tilde{\nabla}$ is orthogonal, torsion-free.

Def $\gamma: [0,T] \to M$ is geodesic if $(\gamma^*\nabla)_{\frac{\partial}{\partial t}}(\gamma^*\dot{\gamma}) = 0$, i.e. $\ddot{\gamma}^j + \Gamma^j_{ik}\dot{\gamma}^i\dot{\gamma}^k = 0$

(sometimes also written $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$, using the fact that this will be indep of how we extend $\dot{\gamma}$ to the full M)

Prop Geodesics are local extrema of length (under variation w/ fixed endpoints)

Pf Computation in last lecture. $\left[\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = -\int_0^T \langle V, \nabla_t \dot{\gamma}\rangle\, dt\right]$

Prop For any $v \in T_pM$, there exists $\varepsilon > 0$ and a geodesic $\gamma: [0,\varepsilon] \to M$ with $\gamma(0) = p$, $\dot{\gamma}(0) = v$

Any two such geodesics agree on their common domain.

Pf Existence + uniqueness theorem for nonlinear 1^{st}-order DE.

Let γ_v denote the geodesic $\gamma_v: [0,T) \to M$ with $\gamma(0) = p$, $\dot{\gamma}(0) = v$, and T maximal. (maybe $T = \infty$).

Prop For any $v \in T_pM$ and $c, t \in \mathbb{R}$, $\gamma_{cv}(t) = \gamma_v(ct)$.

Pf Just check $t \mapsto \gamma_v(ct)$ is indeed a geodesic w/ initial vector cv.

Def Let $D = \{v \in TM : \gamma_v(t) \text{ is defined at } t=1\}$.

Exponential map is $\quad \exp : D \to M$
$\qquad\qquad\qquad\qquad\quad v \mapsto \gamma_v(1)$

Prop a) $D \subset TM$ is open, each $D_p \subset T_pM$ star-shaped

b) $\gamma_v(t) = \exp(tv)$ when both sides defined

c) $\exp$ is smooth.

Pf b) is easy. For a), c): interpret geodesics as integral curves of a global vector field G on TM.

Local coords x^i on $M \rightsquigarrow$ coords (x^i, v^i) on TM; $\quad G = v^k \frac{\partial}{\partial x^k} - v^i v^j \Gamma_{ij}^k(x) \frac{\partial}{\partial v^k}$

Then flow along G is $\frac{dx^k}{dt} = G(dx^k) = v^k$, $\quad \frac{dv^k}{dt} = G(dv^k) = -v^i v^j \Gamma_{ij}^k(x)$.

So $\exp$ is the time-1 flow of G, composed with $\pi : TM \to M$.

But time-1 flow of a smooth vector field is smooth (ODE thm: smooth dependence on initial cond);

the locus where time-1 flow exists is open; and π is an open mapping. ■

Prop Isometries take geodesics to geodesics, ie $\varphi : M \to \tilde{M}$ iso $\Rightarrow \exp(t\varphi_* v) = \varphi(\exp tv)$

Pf Use $\varphi^* \tilde{\nabla} = \nabla$.

Ex Geodesics in $\mathbb{R}^n$ are straight lines. (Pf $\Gamma_{ij}^k = 0$ so geodesic eq. becomes $\ddot{\gamma}^i = 0$)

$\gamma_v(t) = p + tv \qquad \exp((p,v)) = p + v \qquad D = TM$

Ex Geodesics in S^n are great circles, i.e. intersections between S^n and 2-planes in $\mathbb{R}^{n+1}$.

(e.g. geodesics thru south pole are straight lines thru 0 in the stereographic coordinate u.)

Pf Could compute directly using formula for Γ.

But, easier to use isometry group $O(n+1)$.

In pt, consider $\begin{array}{l} p = (1,0,\dots,0) \\ v = (0,1,\dots,0) \end{array}$ 

There is an isometry $\varphi : (x^1, \dots, x^{n+1}) \mapsto (x^1, x^2, -x^3, \dots, -x^{n+1})$ with $\varphi_* v = v$.

Thus $\varphi(\exp tv) = \exp t\varphi_* v = \exp tv$.

So $\exp tv$ lies in $S^n \cap \mathbb{R}^2$. From here it's easy to see it is a great circle with unit speed parameterization.

Get all other geodesics by acting on this one by isometries.

Ex Geodesics in $\mathbb{H}^n$ are <u>great hyperbolas</u>, i.e. intersections between $\mathbb{H}^n$ and 2-planes in $\mathbb{R}^{n,1}$.

Pf Similar to above.