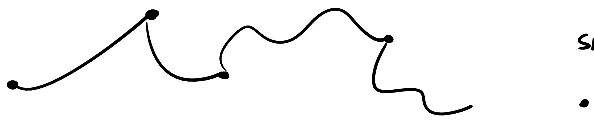
Riemannian distance

(M,g) Riem mfd $\quad p,p' \in M$

Def an admissible curve in M is a map $\gamma : [0,T] \to M$ which is piecewise immersion (divided into finitely many segments; smooth w/ $\dot\gamma \neq 0$ on each segment)



Def length $L(\gamma)$ of an admissible curve is $\sum$ of lengths of the segments $\int_a^b \|\dot\gamma(t)\|\, dt$

Def For $p,p' \in M$, $d(p,p') = \inf\limits_{\substack{\gamma \text{ admissible} \\ \gamma(0)=p,\ \gamma(T)=p'}} L(\gamma)$

Lemma $d(p,p')$ makes M a metric space, inducing the usual topology on M.

Pf $d(p,p') \geq 0$ easy

Δ ineq. easy

Need to show that $d(p,p') > 0$ for $p \neq p'$.

For this, pick nbhd U of p, and coords w/ $g_{ij}(p) = \delta_{ij}$. Shrinking U if necessary we can arrange $p' \notin U$.

Let $K = \{(q,v) \in TM : q \in \bar U,\ \|v\| = 1\}$. K compact. Then define $c = \min\limits_{v \in K} \frac{\|v\|_g}{\|v\|_h}$.

Then $\|v\|_g > c\|v\|_h$ where $h_{ij} = \delta_{ij}$. U contains some coordinate ε-ball around p.

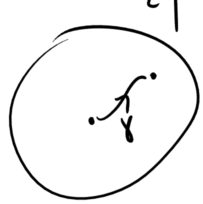
Thus any path which exits U has length $\geq \varepsilon$ in the metric h. $\left(\int \|\dot\gamma\|_h\, ds \geq \int |\dot\gamma_1|\, ds \geq \int \dot\gamma_1\, ds = \varepsilon\right)$

& " " " " " " " $\geq c\cdot\varepsilon$ " " " g.

So $d(p,p') > c\varepsilon$.

To compare topologies: fix a coordinate ball $B_\varepsilon(p) = B_{\varepsilon,h}(p)$. On this ball $c\|v\|_h < \|v\|_g$

so $B_{\varepsilon,h}(p) \supset B_{c\varepsilon,g}(p)$



$L_g(\gamma) < c\varepsilon \Rightarrow L_h(\gamma) < \frac{1}{c} L_g(\gamma) < \varepsilon$

similarly in the other direction.