Curvature for Riemannian submanifolds

Motivation:
- Show that some Riem. metrics are not attained by hypersurfaces in $\mathbb{R}^n$
- Understand the following. Consider a surface S in $\mathbb{R}^3$.

Fix $p \in S$ and a normal direction at p. For every $v \in T_pM$ let $\kappa(v)$ be $\pm\frac{1}{R}$

where R = radius of the osculating circle to a curve with tangent vector v, $\pm$ determined by whether the circle bends toward or away from the chosen normal direction.

Let κ_1 and κ_2 be the max and min $\kappa(v)$, $v \in T_pM$.

Then, $S(p) = 2\kappa_1\kappa_2$. (and κ_1, κ_2 are attained in orthogonal directions)

In p'tc, $\kappa_1\kappa_2$ is intrinsic to the Riem geometry of S!

[Applicable to eating pizza: $S = 0$ so either κ_1 or κ_2 must vanish; can force $\kappa_1 \neq 0$ so then $\kappa_2 = 0$ necessarily.]

Second fundamental form

Say we have $M \subset \widetilde{M}$ immersed Riemannian. Let $\mathcal{N}(M) = \mathcal{E}(NM)$.

Then for $X, Y \in \mathcal{T}(M)$ have
$$\widetilde{\nabla}_X Y = \underset{\mathcal{T}(M)}{\underset{\cap}{(\widetilde{\nabla}_X Y)^{\top}}} + \underset{\mathcal{N}(M)}{\underset{\cap}{(\widetilde{\nabla}_X Y)^{\perp}}}$$

Define $\mathrm{II}(X,Y) = (\widetilde{\nabla}_X Y)^{\perp}$ ("second fundamental form")

Lemma
1) $\mathrm{II} \in \mathcal{E}(\mathrm{Hom}(TM \otimes TM, NM))$
2) $\mathrm{II}(X,Y) = \mathrm{II}(Y,X)$

Pf 1) Check $C^\infty(M)$-linearity:
$$\left[\widetilde{\nabla}_X (fY)\right]^{\perp} = \left[f\widetilde{\nabla}_X Y + (Xf)Y\right]^{\perp} = f\left[\widetilde{\nabla}_X Y\right]^{\perp}$$

2) $\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X = [X,Y] \in TM$

so $(\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X)^{\perp} = 0$.

Lemma $(\tilde{\nabla}_X Y)^T = \nabla_X Y.$

Pf Exercise. (already assigned in the case $M = \mathbb{R}^n$)

Lemma If $X, Y \in TM$ and $N \in NM$ then $\langle \tilde{\nabla}_X N, Y \rangle = -\langle N, \mathrm{II}(X,Y) \rangle$

Pf
$$\begin{aligned} 0 &= X \langle N, Y \rangle \\ &= \langle \tilde{\nabla}_X N, Y \rangle + \langle N, \tilde{\nabla}_X Y \rangle \\ &= \langle \tilde{\nabla}_X N, Y \rangle + \langle N, \nabla_X Y + \mathrm{II}(X,Y) \rangle \\ &= \langle \tilde{\nabla}_X N, Y \rangle + \langle N, \mathrm{II}(X,Y) \rangle \end{aligned}$$

Now we can relate curvatures of M and $\tilde{M}$:

Thm $\widetilde{Rm}(X,Y,Z,W) = Rm(X,Y,Z,W) - \langle \mathrm{II}(X,W), \mathrm{II}(Y,Z) \rangle + \langle \mathrm{II}(X,Z), \mathrm{II}(Y,W) \rangle$

Pf
$$\begin{aligned} \widetilde{Rm}(X,Y,Z,W) &= \langle \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z, W \rangle \\ &= \langle \tilde{\nabla}_X (\nabla_Y Z + \mathrm{II}(Y,Z)) - \tilde{\nabla}_Y (\nabla_X Z + \mathrm{II}(X,Z)) - \tilde{\nabla}_{[X,Y]} Z + \mathrm{II}([X,Y],Z), W \rangle \\ &= \langle \tilde{\nabla}_X \nabla_Y Z, W \rangle - \langle \mathrm{II}(Y,Z), \mathrm{II}(X,W) \rangle - \langle \tilde{\nabla}_Y \nabla_X Z, W \rangle + \langle \mathrm{II}(Y,W), \mathrm{II}(X,Z) \rangle \\ &\quad - \langle \tilde{\nabla}_{[X,Y]} Z, W \rangle \qquad \text{using previous Lemma} \\ &= \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W \rangle - \langle \mathrm{II}(X,W), \mathrm{II}(Y,Z) \rangle + \langle \mathrm{II}(Y,W), \mathrm{II}(X,Z) \rangle \end{aligned}$$

using $\langle \tilde{\nabla}(\cdot\cdot), W \rangle = \langle \nabla(\text{--}), W \rangle$

Rk Say $\gamma : [0,T] \to M$ and X a vector field along γ.

Then $\tilde{\nabla}_{\dot\gamma} X = \nabla_{\dot\gamma} X + \mathrm{II}(\dot\gamma, X)$.

In ptc, if γ is a geodesic in M then $\tilde{\nabla}_{\dot\gamma} \dot\gamma = \mathrm{II}(\dot\gamma, \dot\gamma)$

So II measures the acceleration in $\tilde{M}$ of geodesics in M.

(If $\mathrm{II} = 0$, say M is totally geodesic in $\tilde{M}$.)

Now, specialize to hypersurfaces $M \subset \mathbb{R}^n$.

Pick a unit normal vector N to M at p, then define $s: T_pM \to T_pM$ ("shape tensor")
by $\langle s(X), Y\rangle = \langle \mathrm{II}(X,Y), N\rangle$.

<u>Def</u> The <u>Gaussian curvature</u> of M at p is $K(p) = \det s$.
The <u>mean curvature</u> of M at p is $m(p) = \operatorname{tr} s$.

[<u>Rk</u> $m = 0 \iff M \subset \mathbb{R}^n$ is locally <u>volume-minimizing</u>.]

In general $K(p)$, $m(p)$ <u>depend on</u> the embedding in $\mathbb{R}^n$. But for $n=3$ there is a miracle:

<u>Thm</u> (Gauss) If $M \subset \mathbb{R}^3$ surface, then

1) in an ON-basis, $K = R_{1221}$

2) for $X, Y \in T_pM$, $K = \dfrac{Rm(X,Y,Y,X)}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}$

<u>Pf</u> 1) Since $\mathbb{R}^n$ is <u>flat</u>, we have

$$R_{1221} = Rm(e_1,e_2,e_2,e_1) = \langle \mathrm{II}(e_1,e_1), \mathrm{II}(e_2,e_2)\rangle - \langle \mathrm{II}(e_1,e_2), \mathrm{II}(e_2,e_1)\rangle$$
$$= \det\left(\langle s(e_i), e_j\rangle\right)_{i,j=1,2}$$
$$= \det s$$

1) $\Rightarrow$ 2) is linear algebra exercise (do Gram-Schmidt to make ON basis from X, Y and then compute R_{1221})

<u>Cor</u> If $M \subset \mathbb{R}^3$ surface, $S(p) = 2K(p) = 2\kappa_1\kappa_2$.

<u>Pf</u> In ON-basis, $S = (Ric)_{11} + (Ric)_{22} = R_{1221} + R_{2112} = 2R_{1221} = 2K$.

Let $f(v) = \dfrac{\langle v, s(v)\rangle}{\|v\|^2} = \dfrac{\mathrm{II}(v,v)}{\|v\|^2}$. $f(v)$ is the inverse curv. of osculating circle to v.
(from high school: $a = \frac{v^2}{r}$!)

s is self-adjoint $\Rightarrow$ $f(v)$ is extremized at <u>eigenvectors</u> of v.

$s(v) = \kappa v \Rightarrow f(v) = \kappa$. So $K = \det s = \kappa_1\kappa_2$. ✓