The Hodge star

M oriented Riem mfd.

Def/Prop $\star : \Omega^p(M) \to \Omega^{n-p}(M)$ is determined by $\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \, \mathrm{vol}$.

Ex If $n=3$, in ON-basis $\{e_i\}$,

$$\mathrm{vol} = e_1 \wedge e_2 \wedge e_3$$

$$\star(1) = e_1 \wedge e_2 \wedge e_3 \qquad \star(e_1 \wedge e_2) = e_3$$

$$\star(e_1) = e_2 \wedge e_3 \qquad \star(e_2 \wedge e_3) = e_1$$

$$\star(e_2) = e_3 \wedge e_1 \qquad \star(e_3 \wedge e_1) = e_2$$

$$\star(e_3) = e_1 \wedge e_2 \qquad \star(e_1 \wedge e_2 \wedge e_3) = 1$$

Prop $\star^2 = (-1)^{p(n-p)}$ acting on $\Omega^p(M)$

Rk Reversing orientation of M takes $\star \to -\star$.

The p-form Laplacian

M oriented Riem manifold:

$$d : \Omega^p(M) \to \Omega^{p+1}(M)$$

Def 1) $d^* : \Omega^p(M) \to \Omega^{p-1}(M)$ given by $d^*\omega = (-1)^{n(p+1)+1} \star d \star \omega$

uses two $\star$, so $\exists$ even for M not oriented!

2) $\Delta : \Omega^p(M) \to \Omega^p(M)$ given by $\Delta = dd^* + d^*d$

Prop If M compact, for L^2 pairing $\langle \alpha, d^*\beta \rangle_{L^2} = \langle d\alpha, \beta \rangle_{L^2}$

Pf $\langle \alpha, \beta \rangle_{L^2} = \int \langle \alpha, \beta \rangle \, \mathrm{vol} = \int \alpha \wedge \star\beta$

so
$$\begin{aligned}
\langle d\alpha, \beta \rangle_{L^2} &= \int d\alpha \wedge \star\beta \\
&= (-1)^{1+|\alpha|} \int \alpha \wedge d\star\beta \\
&= (-1)^{1+|\alpha|+|\alpha|(n-|\alpha|)} \int \alpha \wedge \star(\star d \star \beta) \\
&= \langle \alpha, d^*\beta \rangle_{L^2}
\end{aligned}$$

[since $1 + |\alpha| + |\alpha|(n-|\alpha|) = n(|\beta|+1)+1 \mod 2$
using $|\alpha| + 1 = |\beta|$]

Thus we call d^* a "formal adjoint" to d.

Cor If M compact, $\langle \alpha, \Delta\beta \rangle_{L^2} = \langle df, dg \rangle_{L^2} + \langle d^*f, d^*g \rangle_{L^2} = \langle \Delta\alpha, \beta \rangle_{L^2}$.

Cor If M compact, $\Delta\alpha = \lambda\alpha$, then $\lambda \geq 0$; if $\lambda = 0$ then $d\alpha = 0$, $d^*\alpha = 0$.

Pf $\lambda \|\alpha\|_{L^2}^2 = \langle \alpha, \Delta\alpha \rangle_{L^2} = \|d\alpha\|_{L^2}^2 + \|d^*\alpha\|_{L^2}^2$

[**Rk** This really needs M compact — e.g. if $M = \mathbb{R}$ and $f(x) = e^x$, $\Delta f = -f$.]

Def $\mathcal{H}^p(M) = \ker(\Delta : \Omega^p(M) \to \Omega^p(M))$

Cor $\dim \mathcal{H}^0(M) = \#$ connected components of $M = b^0(M)$.

This fact has an important refinement:

Def (de Rham cohomology) M smooth manifold:
$$H^p_{dR}(M) = \frac{\ker(d : \Omega^p(M) \to \Omega^{p+1}(M))}{\operatorname{im}(d : \Omega^{p-1}(M) \to \Omega^p(M))}$$
$$b^p(M) = \dim_{\mathbb{R}} H^p(M)$$

Ex $H^0_{dR}(M) = \{\text{locally constant functions}\}$ so $b^0(M) = \#$ connected components of M

Ex $M = S^1$: $H^1_{dR}(M) = \dfrac{\Omega^1(M)}{\{df\}}$.

$$\omega = df \implies \int_{S^1} \omega = 0 \text{ by Stokes.}$$

If $\int_{S^1} \omega = 0$ then $\exists f$ s.t. $\omega = df$, namely $f(x) = \int_{[0,x]} \omega$.

So, have map
$$H^1_{dR}(M) \xrightarrow{\sim} \mathbb{R}$$
$$\omega \longmapsto \int_{S^1} \omega$$
and thus $b^1(M) = 1$

But, it seems we have no canonical representative in each class: e.g. no preferred ω with $\int_{S^1} \omega = 1$.

Rk $H^p_{dR}(M) \simeq H^p_{sing}(M, \mathbb{R})$.

So this is another way of thinking about the "usual" cohomology of M.

Thm (Hodge) If M compact,

Then each class in $H^p_{dR}(M)$ contains a unique element of $\mathcal{H}^p(M)$

Rk Note $H^p_{dR}(M)$ is defined without a metric, while $\mathcal{H}^p(M)$ depends on one a priori.

Pf Sketch If $\omega \in \mathcal{H}^p(M)$ then $d\omega = 0$, so have a map $\mathcal{H}^p(M) \longrightarrow H^p_{dR}(M)$.

- Injective: suppose $\omega \in \mathcal{H}^p(M)$, $\omega = d\alpha$; then $\|\omega\|^2 = \langle \omega, d\alpha \rangle = \langle d^*\omega, \alpha \rangle = 0$.
- Surjective: first note $\operatorname{Im} d$, $\operatorname{Im} d^*$, and $\mathcal{H}^p$ are all mutually orthogonal.

  Suppose we knew $\Omega^p = d\Omega^{p-1} \oplus d^*\Omega^{p+1} \oplus \mathcal{H}^p$. (see below)

  Then, given γ with $d\gamma = 0$, $\gamma = d\alpha + d^*\beta + \delta$ $\quad \delta \in \mathcal{H}^p$

  $$d\gamma = dd^*\beta = 0$$

  but then $\langle \beta, dd^*\beta \rangle_{L^2} = 0$ so $\|d^*\beta\|^2 = 0$, i.e. $d^*\beta = 0$.

  So, $\gamma = d\alpha + \delta$.

  But then $[\gamma] = [\delta]$ in H^p.

So, what we need is to prove

Lemma $\Omega^p = d\Omega^{p-1} \oplus d^*\Omega^{p+1} \oplus \mathcal{H}^p$.

Pf Sketch It would be enough to show $\Omega^p = \Delta\Omega^p \oplus \mathcal{H}^p$.

(since $\operatorname{Im}\Delta \subset \operatorname{Im} d \oplus \operatorname{Im} d^*$)

Note this would be easy in finite-dimensional setting: just diagonalize Δ to see $\exists\, G : \Omega^p \to \Omega^p$ s.t. for $\omega \in (\mathcal{H}^p)^\perp$, $(\Delta \circ G)\omega = \omega$.

(So in pt. $\omega \in \operatorname{Im}\Delta$.)

We'll discuss how to do it in our ∞-dimensional setting a little later.

Rk Poincaré duality is "easy" in de Rham context: $\Delta \circ \star = \star \circ \Delta$ [Exercise]

and thus $\star : \mathcal{H}^p(M) \xrightarrow{\sim} \mathcal{H}^{n-p}(M)$; so $b_p = b_{n-p}$

Better: $\varphi : H^p(M) \times H^{n-p}(M) \longrightarrow \mathbb{R}$ $\left(\text{well def} - \int_M d\gamma \wedge \beta = \int_M d(\gamma \wedge \beta) = 0\right)$

$$([\alpha], [\beta]) \longmapsto \int_M \alpha \wedge \beta$$

Prop φ is a nondegenerate pairing.

Pf φ degenerate $\Longleftrightarrow \exists\, \alpha \neq 0$ s.t. $\varphi(\alpha, \cdot) = 0$.

But $\varphi(\alpha, \star\alpha) = \|\alpha\|^2$.

Rk Similarly Künneth: $M = M_1 \times M_2$ $\quad \mathcal{H}^{p_1}(M_1) \times \mathcal{H}^{p_2}(M_2) \xrightarrow{\sim} \mathcal{H}^{p_1+p_2}(M_1 \times M_2)$

$\Delta = \Delta_1 + \Delta_2$ $\quad (\alpha, \beta) \mapsto \alpha \wedge \beta$

Rk Warning: α, β harmonic $\not\Rightarrow$ $\alpha \wedge \beta$ harmonic!

So $\wedge$ does not reproduce the "cup product".