Bochner method

Prop If M compact and
- $\mathrm{Ric}(X,X) > 0 \ \forall \overset{\neq 0}{X} \in TM$, then $b_1(X) = 0$.
- $\mathrm{Ric}(X,X) \geq 0 \ \forall \overset{\neq 0}{X} \in TM$, then $b_1(X) \leq \dim X$.

Pf Say $\omega \in \mathcal{H}^1(M)$.

Then $0 = \nabla^* \nabla \omega + \hat{R}\omega$

so $$0 = \langle \omega, \nabla^*\nabla\omega\rangle_{L^2} + \langle \omega, \hat{R}\omega\rangle_{L^2}$$
$$= \underbrace{\|\nabla\omega\|^2_{L^2}}_{\geq 0} + \int_M \underbrace{\mathrm{Ric}(\omega^\#, \omega^\#)}_{\geq 0} \cdot \mathrm{vol}$$

So $\nabla\omega = 0$ and $\mathrm{Ric}(\omega^\#, \omega^\#) = 0$.

But $\dim\{\omega : \nabla\omega = 0\} \leq n$.

$$\left[\begin{aligned} \langle \omega, \hat{R}\omega\rangle &= \langle \omega, R_{ijkl}\, a_i^* a_j a_k^* a_l \omega\rangle \\ &= \langle a_i\omega, R_{ijkl}\, a_j a_k^*(\omega_l)\rangle \\ &= \langle \omega_i, R_{ijkl}\,\delta_{jk}\,\omega_l\rangle \\ &= \langle \omega_i, (\mathrm{Ric})_{il}\,\omega_l\rangle \end{aligned}\right]$$

Ex $n=2$: sphere has a positive curvature metric $\Rightarrow b_1 = 0$ ✓

torus has zero curvature metric $\Rightarrow b_1 \leq 2$ ✓

In general, an <u>n-torus</u> $(S^1)^n$ has $b_1 = n$. So the bound in the thm is sharp.