Connections in principal bundles

Recall that if E is any bundle over M, and we fix a local basis $\{e_i\}$, a connection ∇ can be represented by 1-forms A^j_i characterized by $\nabla e_i = A^j_i e_j$

Under a change of basis $e_{i'} = C^j_{i'} e_j$ we have $A^{j'}_{i'} = (C^{-1})^{j'}_j A^j_i C^i_{i'} + (C^{-1})^{j'}_j dC^j_{i'}$

Less indexy version: $A \in \mathcal{E}(T^* \otimes \mathrm{End}(\mathbb{R}^n))$ with $A' = C^{-1}AC + C^{-1}dC$

Letting $G = GL(n,\mathbb{R})$, $\mathfrak{g} = \mathrm{End}(\mathbb{R}^n)$, so this is also $A \in \mathcal{E}(T^*M \otimes \mathfrak{g})$.

Now we want to give a global interpretation of this formula. Let P be the bundle of frames in E.
Our basis $\{e_i\}$ then $\longleftrightarrow$ a section s of P. $s: M \hookrightarrow P$.
Want to find a single 1-form $\omega \in \mathcal{E}(T^*P \otimes \mathfrak{g})$ such that $A = s^*\omega$.
Transformation law above then relates $s^*\omega$ and $(s\hat{C})^*\omega = s^*(\hat{C}^*\omega)$, for $C: M \to G$ inducing $\hat{C}: P \to P$.
It says, $\hat{C}^*\omega = C^{-1}\omega C + \pi^*[C^{-1}dC]$
Thus:

Def/Prop P principal $GL(n,\mathbb{R})$-bundle:

A connection in P is a 1-form $\omega \in \mathcal{E}(T^*P \otimes \mathfrak{g})$ such that

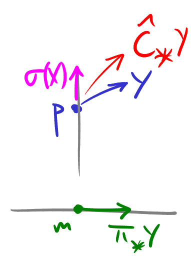
1) $\forall C: M \to G$, $\hat{C}^*\omega = C^{-1}\omega C + \pi^*[C^{-1}dC]$

or equivalently,

2a) $\omega(g_* Y) = \mathrm{Ad}(g^{-1})\,\omega(Y) \quad \forall g \in G,\ Y \in TP$,

2b) $\omega(\sigma(X)) = X \quad \forall X \in \mathfrak{g}$.

Pf 1) $\Rightarrow$ 2a): take C constant.



1) $\Rightarrow$ 2b): Fix $Y \in T_pP$ with $\pi(p) = m$, $\pi_* Y \neq 0$. Fix $X \in \mathfrak{g}$.
Consider $C: M \to G$ with $C(m) = 1$, $dC(\pi_* Y) = X$.
Then, $\hat{C}_* Y - Y = [\sigma(X)](p)$ (see below for comput^n of $\hat{C}_*$)
But $(\hat{C}^*\omega)(Y) - \omega(Y) = (\pi^* dC)(Y)$ by 1),
ie $\omega(\hat{C}_* Y - Y) = X$.

$2) \Rightarrow 1)$:

$$\begin{tikzcd} P \arrow[r, "(1,\pi)"] \arrow[rrr, bend right, "\hat{C}"'] & P\times M \arrow[r, "1_P\times C"] & P\times G \arrow[r, "\cdot"] & P \end{tikzcd}$$

$d(1,\pi): TP \to TP\times TM$
$\quad v \mapsto (v, \pi_* v)$

$d(1_P\times C): TP\times TM \to TP\times TG$
$\quad (v,w) \to v + (dC)(w)$

$d(\cdot): T_pP\times T_cG \to T_{pc}P$
$\quad (v,u) \mapsto C_* v + \sigma(C_*^{-1}u)$

So $\omega(d\hat{C}(v)) = \omega\big(C_* v + \sigma(C_*^{-1} dC(\pi_* v))\big)$
$\qquad = C^{-1}\omega(v)\,C + C^{-1}dC(\pi_* v)$

ie $\quad \hat{C}^*\omega = C^{-1}\omega C + \pi^*(C^{-1}dC)$ as desired.

In fact, our second def^n made sense for any G:

Def P principal G-bundle:

A connection in P is a 1-form $\omega \in \mathcal{E}(T^*P\otimes\mathfrak{g})$ such that

$\omega(g_* Y) = Ad(g^{-1})\,\omega(Y) \quad \forall g\in G,\ Y\in TP,$
$\omega(\sigma(X)) = X \quad \forall X\in\mathfrak{g}.$

Def $T^{vert}P = \ker \pi_*$. Given a conn. in P, define $T^{horiz}P = \ker\omega$.

Prop 1) $TP = T^{vert}P \oplus T^{horiz}P$.

2) $g_*(T^{horiz}P) = T^{horiz}P$.

3) Any distribution $T^{horiz}P$ obeying 1), 2) determines a connection in P.

Rk Given a connection in P we also obtain horiz. distrib. on all Y_P.

Def $O(n) = \{M\in GL(n): M^TM = 1\}$, $SO(n) = \{M \in O(n): \det M = 1\}$.

Ex $G = SO(2)$: P is a circle bundle over M. $\Rightarrow$ wrt a local triv of P, have coords (x^i, θ).

Then identifying $\mathfrak{g}$ with $\mathbb{R}$, ω is just a 1-form,

$\omega = \omega_0 + d\theta \qquad$ where $\omega_0(\partial/\partial\theta) = 0$.

Pulling back by the section $\theta = 0$ gives $A = \omega_0$ ⟵ related by gauge xform
" " " $\theta = f(x)$ " $A = \omega_0 + df$ ⟵

Rk If E is an orthogonal v.b. we can also consider the bundle of <u>orthonormal</u> <u>frames</u> in E,
$P_x = \{\text{orthonormal bases for } E_x\}$. P_x is a principal $O(n)$-bundle.

Rk Given a homomorphism $\rho: G \to H$, G acts on H (by $g \cdot h = \rho(g) h$)
Then if P principal G-bundle, H acts on $P \times_G H$ by $[(p,h)] \mapsto [(p, hh')]$
This makes $P \times_G H$ into a principal H-bundle.
If Q is an H-bundle realized as $P \times_G H$ for some $G \subset H$, say Q "reduces to G".

Ex Say Q is bundle of frames in some E. Fix a metric on E. Then let P be bundle of orth. frames in E. $P \times_{O(n)} GL(n) \simeq Q$ [Exercise].
Thus Q reduces to $O(n)$.
But in general, given $H \subset G$, there are topological obstructions to a G-bundle reducing to H.
(e.g. consider $H = \{1\}$)

<u>Curvature</u>

Given principal G-bundle $P \to M$ and conn. ω in P,
define its <u>curvature</u> $\Omega = d\omega + \omega \wedge \omega \in \mathcal{E}(\Lambda^2 T^* P \otimes \mathfrak{g})$

ie $\quad \Omega(Y,Z) = d\omega(Y,Z) + [\omega(Y), \omega(Z)] \qquad Y,Z \in TP$

Prop Ω obeys
1) $\Omega(g_* Y, g_* Z) = Ad(g^{-1}) \Omega(Y,Z) \quad \forall g \in G,\ Y,Z \in TP$,
2) $\Omega(Y,Z) = 0 \quad \forall Y \in (TP)_{vert},\ Z \in TP$

Pf
1) from the analogous property for ω.
2) $\Omega(Y,Z) = Y\omega(Z) - Z\omega(Y) - \omega([Y,Z]) + [\omega(Y), \omega(Z)]$
If Y,Z both vertical then extend them as $Y = \sigma(W)$, $Z = \sigma(X)$ for $W, X \in \mathfrak{g}$:

then $\Omega(Y,Z) = \underbrace{\sigma(W)\cdot X}_{0 \text{ b/c } X \text{ const.}} - \underbrace{\sigma(X)\cdot W}_{0 \text{ b/c } W \text{ const.}} - \omega([\sigma(W),\sigma(X)]) + [W,X]$

$$= -[W,X] + [W,X] = 0 \qquad \left(\text{using } \underset{\text{[Exercise]}}{[\sigma(W),\sigma(X)] = \sigma([W,X])}\right)$$

(This is essentially Maurer–Cartan equation)

If Y vertical and Z horizontal, then extend $Y = \sigma(W)$ and Z as section of $\ker\omega$:

then $\Omega(Y,Z) = \underbrace{\sigma(W)\cdot\omega(Z)}_{0 \text{ b/c } \omega(Z)=0} - \underbrace{Z\cdot W}_{0 \text{ b/c } W \text{ const.}} - \underbrace{\omega([\sigma(W),Z])}_{\substack{0 \text{ b/c} \\ [\sigma(W),Z] \text{ horiz.} \\ \text{[Exercise]}}} + \underbrace{[W,\omega(Z)]}_{0 \text{ b/c } \omega(Z)=0}$

Cor If s, s' are two sections of P with $s' = sg$ for $g: M \to G$,
then $s'^*\Omega = \mathrm{Ad}(g^{-1})\, s^*\Omega$

Pf Just as we did for ω, using the preceding prop.

Rk $\mathfrak{g}_P = P \times \mathfrak{g} \big/ (P,X) \sim (Pg, \mathrm{Ad}(g^{-1})X)$

Ω induces a section $F \in \mathcal{E}(\Lambda^2 T^* \otimes \mathfrak{g}_P)$, by $F(X,Y) = [(p, \Omega(\hat{X},\hat{Y}))]$
for any $\hat{X},\hat{Y} \in T_pP$, with $\pi_*\hat{X} = X$, $\pi_*\hat{Y} = Y$

Fix any rep $\rho: G \to \mathrm{Aut}(V)$.

Prop 1) ω induces a connection ∇ in V_ρ, by $\nabla[(s,v)] = [(s, dv + (s^*\omega)v)]$

2) Its curvature $F_\nabla \in \mathcal{E}(\Lambda^2 T^* \otimes \mathrm{End}\, V_\rho)$ is $F_\nabla(X,Y)[(s,v)] = [(s, \Omega(\hat{X},\hat{Y})v)]$
for any $\hat{X},\hat{Y} \in T_sP$ with $\pi_*\hat{X} = X$, $\pi_*\hat{Y} = Y$.

Pf Exercise.

Rk Conversely, if ρ is faithful, a conn. in V_ρ induces one in P.