Gauss-Bonnet-Chern

Prop If $G = SO(n)$, $\mathfrak{g} = \mathfrak{so}(n) \subset \overset{\overset{G}{\curvearrowright}}{\mathbb{R}^n} \otimes \overset{\overset{G}{\curvearrowright}}{(\mathbb{R}^n)^*}$. Using the metric to identify $(\mathbb{R}^n)^* \simeq \mathbb{R}^n$ gives $\varphi : \underset{\circlearrowleft G}{\mathbb{R}^n \otimes (\mathbb{R}^n)^*} \xrightarrow{\sim} \underset{\circlearrowleft G}{\mathbb{R}^n \otimes \mathbb{R}^n}$. Then $\varphi(\mathfrak{g}) = \Lambda^2(\mathbb{R}^n)$.

Pf Say $X \in \mathfrak{so}(n)$. Then take $g(t)$ with $g(0) = 1$, $g'(0) = X$.

$v, w \in \mathbb{R}^n$: $\langle v, w \rangle = \langle g(t)v, g(t)w \rangle$

$$\frac{d}{dt}\Big|_{t=0} \Big[\langle g(t)v, g(t)w \rangle \Big] = 0$$

$$\langle v, Xw \rangle + \langle Xv, w \rangle = 0$$

Thus we have a G-invariant identification $\underset{\underset{G}{\circlearrowright}}{\mathfrak{g}} \simeq \underset{\underset{G}{\circlearrowleft}}{\Lambda^2(\mathbb{R}^n)}$.

Def If $n = 2m$, $X \in \mathfrak{g} \simeq \Lambda^2(\mathbb{R}^{2m})$,

$$Pf(X) = (\underbrace{X \wedge \cdots \wedge X}_{m}) / vol \qquad Pf : \mathfrak{g} \longrightarrow \mathbb{R}$$

$$Pf(X_1, \dots, X_m) = (X_1 \wedge \cdots \wedge X_m) / vol \qquad Pf : \mathfrak{g}^{\otimes m} \to \mathbb{R}$$

Prop $Pf(Ad_g X) = Pf(X)$.

Pf All our constructions were G-equivariant, so we got a G-equiv't map $\mathfrak{g} \to \mathbb{R}$.

Rk $[Pf(X)]^2 = \det(X)$.

Ex $m = 1$: $X = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, $\det(X) = a^2$, $Pf(X) = a$.

For general m, use canonical form e.g. $X \sim X' = \begin{pmatrix} 0 & a & & \\ -a & 0 & & \\ & & 0 & b \\ & & -b & 0 \end{pmatrix}$, $\det(X') = a^2 b^2$, $Pf(X') = ab$

(cf. a general matrix: usually $\sqrt{\det X}$ would not be a polynomial in entries of X..)

Now, say M is a Riem mfd of dim $n = 2m$.

Have Riemann curvature $R \in \mathcal{E}(\Lambda^2 T^*M \otimes \mathfrak{g}_P)$ $\qquad \mathfrak{g} = so(2m)$

Def $Pf : [\mathcal{E}(\Lambda^2 T^*M \otimes so(2m))]^{\otimes m} \longrightarrow \mathcal{E}(\Lambda^{2m} T^*M)$ is determined by

$$Pf(\omega_1 \otimes X_1, \omega_2 \otimes X_2, \dots, \omega_m \otimes X_m) = (\omega_1 \wedge \cdots \wedge \omega_m) Pf(X_1, \dots, X_m)$$

Def/Prop $Pf : \mathcal{E}(\Lambda^2 T^*M \otimes \mathfrak{g}_P) \longrightarrow \mathcal{E}(\Lambda^{2m} T^*M)$ determined by

$$Pf([(s, W)]) = Pf(W, W, \dots, W). \qquad W \in \mathcal{E}(\Lambda^2 T^*M \otimes \mathfrak{g})$$

Pf Well defined: $Pf([(s, W)]) = Pf([(sg, g^{-1}W)])$ by Ad-invariance of Pf.

Ex $m = 1$: $Pf(R) = \frac{1}{2} S \cdot vol$.

Let P = bundle of oriented ON frames of M (principal $SO(n)$-bundle)

Q = unit tangent bundle of $M = \{v \in TM : \|v\| = 1\}$.

Prop $P \times_{SO(n)} S^{n-1} \simeq Q$.

Pf $e \in P_m$ is an orthogonal map $e : \mathbb{R}^n \longrightarrow T_m M$

Then the desired map is $(e, v) \longmapsto e(v)$.

[**Rk** Thus, Q has "preferred trivs": given a section s of P, get a trivializ^n of Q.
Different triv. differ by maps $g : M \longrightarrow SO(n)$.]

Now, suppose given a section Y of TM w/ isolated zeroes. $u = \dfrac{Y}{\|Y\|}$ is a section of Q over $M' = M \setminus \{\text{zeroes of } Y\}$.

Say $Y(m) = 0$. Fix a section of P in nbhd U of m, $Q|_U \simeq U \times S^{n-1}$.

Then fix a sphere $S^{n-1}_\varepsilon \subset U \setminus \{m\}$. u gives a map $u : S^{n-1}_\varepsilon \longrightarrow S^{n-1}$.

Def $\mathrm{ind}_m Y = \left(\int_{S^{n-1}_\varepsilon} u^* vol \right) \Big/ \int_{S^{n-1}} vol$.

Prop $\mathrm{ind}_m Y \in \mathbb{Z}$.

Pf Need a drop of algebraic topology here: $H^{n-1}_{dR}(S^{n-1}) \simeq [H^{n-1}(S^{n-1};\mathbb{Z})] \otimes \mathbb{R}$

Then $u^*: H^{n-1}(S^{n-1};\mathbb{Z}) \longrightarrow H^{n-1}(S^{n-1};\mathbb{Z})$

$\mathrm{vol} \longmapsto k \cdot \mathrm{vol} \qquad k \in \mathbb{Z}$

Thm (Gauss-Bonnet-Chern) $\displaystyle\sum_{m: Y(m)=0} \mathrm{ind}_m Y = \frac{1}{(2\pi)^{n/2}} \int_M \mathrm{Pf}(R)$.

Pf Idea: $\exists \beta \in \mathcal{E}(\Lambda^{2n-1} T^*Q)$ such that:

- $d\beta = \pi^*[\mathrm{Pf}(R)]$
- $\beta\big|_{\pi^{-1}(m)} = (2\pi)^{n/2} \mathrm{vol}$. (vol = fiberwise volume form on Q)

Then, do as with the pf of Gauss-Bonnet:

$$\int_M \mathrm{Pf}(R) = \lim_{\varepsilon\to 0} \int_{M_\varepsilon} \mathrm{Pf}(R) = \lim_{\varepsilon\to 0} \int_{u(M_\varepsilon)} \pi^* \mathrm{Pf}(R) = \lim_{\varepsilon\to 0} \int_{\partial M_\varepsilon} \beta = (2\pi)^{n/2} \sum_{m: Y(m)=0} \mathrm{ind}_m Y$$

So, main part is to construct β. (In case $n=1$ just had $\beta = \omega$.)

We could do this ad hoc as done by Chern. But, can also understand it in a more intrinsic way:

Note $S^n = SO(n+1)/SO(n)$, and use

Lemma Suppose P is a principal G-bundle, $H \subset G$ "reductive" ie $\exists\ \mathfrak{h} \underset{P}{\leftrightarrows} \mathfrak{g}$ Ad_H-inv^t.

Suppose P has a conn. ω with curvature $F \in \mathcal{E}(\Lambda^2 T^*M \otimes \mathfrak{g}_P)$.

Let $Q = P \times_G (G/H) = P/H$.

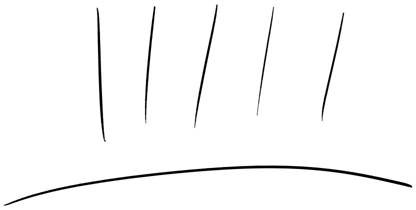
Suppose $\Phi: \mathfrak{g}^{\otimes k} \longrightarrow \mathbb{R}$ Ad-invariant, and $\Phi = 0$ on $\mathfrak{h}^{\otimes k} \subset \mathfrak{g}^{\otimes k}$.

Then, $\exists \beta \in \mathcal{E}(\Lambda^{2k-1} T^*P)$ s.t. $d\beta = \pi^*(\Phi(F))$.

Pf First, consider the special case $H = 1$. (then β will be "Chern-Simons form")

Morally, we just want to "integrate" $\Phi(F)$.
What does that mean?

Recall "relative Stokes":



$$\begin{tikzcd} N \arrow[d] \\ M \end{tikzcd}$$

$\gamma \in \Omega^k(N) \rightsquigarrow \int_N \gamma \in \Omega^{k-m}(M)$

$$d\left[\int_N \gamma\right] = \int_N d\gamma - \int_{\partial N} \gamma$$

So if we have a closed form γ on $I \times M$

then $d\left[\int_I \gamma\right] = \int_{\partial I} \gamma = \gamma(1) - \gamma(0)$

and if also $\gamma(0) = 0$, then $d\left[\int_I \gamma\right] = \gamma(1)$

And $\Phi(F)$ is a closed form [Exercise]

Thus, a possible strategy: find a closed form interpolating between $\Phi(F)$ and zero.

e.g.: find a 1-parameter family of connections, interpolating between ω and a flat connection. If this worked, it would show $\Phi(F)$ is exact

Problem: P might not have any flat connections!

So: pull back. $\pi^*P = \{(p_1,p_2) \in P \times P : \pi(p_1) = \pi(p_2)\}$

$$\begin{tikzcd} & \pi^*P \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"', "{\pi_1^*\omega,\ \omega_0}"] & \\ P \arrow[dr, "\pi_3"', "\omega"] & & P \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

$\pi^*P \xrightarrow{\pi_2} P$ has a tautological section $\Longrightarrow$ a tautological flat connection ω_0. It also has the connection $\pi^*\omega$.

So: now look at $\pi^*P \times I \xrightarrow{\pi_2 \times 1} P \times I$ $\quad I = \{t \in [0,1]\}$

with connection $\underline{\omega} = (1-t)\omega_0 + t\pi_1^*\omega$

Then $\Phi(\underline{\Omega})$ is closed; $d\left[\int_I \Phi(\underline{\Omega})\right] = \Phi(\pi_1^*\Omega) - \Phi(\underline{\Omega}_0)$

$= \Phi(\pi_1^*\Omega)$

and $\exists \beta$ s.t. $\int_I \Phi(\underline{\Omega}) = \pi_2^*\beta$; $\Phi(\pi_1^*\Omega) = \pi_1^*\pi_3^*\Phi(F) = \pi_2^*\pi^*\Phi(F)$

so finally $d\beta = \pi^*\Phi(F)$ as desired.

A basic example: $\Phi(X) = Tr(X^2)$. (For any G, once we fix a rep..)

Then e.g. if P is trivial, $P = M\times G$: $\omega(\cdot, g) = Ad_g(A) + \theta_G$,

$$\pi_1^*\omega(\cdot, g_1, g_2) = Ad_{g_1}(A) + \theta_G^{(1)}$$

$$\omega_o(\cdot, g_1, g_2) = Ad_{g_1}(\theta_G^{(2)}) + \theta_G^{(1)}$$

Thus, $\underline{\omega} = (1-t)\left(Ad_{g_1}(\theta_G^{(2)}) + \theta_G^{(1)}\right) + t\left(Ad_{g_1}(A) + \theta_G^{(1)}\right)$

$$= (1-t)Ad_{g_1}(\theta_G^{(2)}) + t\,Ad_{g_1}(A) + \theta_G^{(1)}$$

$$\underline{\Omega} = d\underline{\omega} + \underline{\omega}\wedge\underline{\omega}$$

$$= Ad_{g_1}\left[dt\wedge A + t\,dA - dt\wedge\theta_G^{(2)} + (-(1-t) + (1-t)^2)\,\theta_G^{(2)}\wedge\theta_G^{(2)} + t^2 A\wedge A + t(1-t)A\wedge\theta_G^{(2)}\right]$$

$$\int_I Tr(\underline{\Omega}\wedge\underline{\Omega}) = 2\int_0^1 Tr\left((A - \theta_G^{(2)})\wedge\left(t\,dA + (t^2 - t)\,\theta_G^{(2)}\wedge\theta_G^{(2)} + t^2 A\wedge A + t(1-t)A\wedge\theta_G^{(2)}\right)\right)$$

$$= 2\,Tr\left(\tfrac{1}{2}A\wedge dA + \tfrac{2}{3}A\wedge A\wedge A - \tfrac{1}{6}\theta_G\wedge\theta_G\wedge\theta_G + \text{mixed terms}\right)$$

[AKA "Chern-Simons action" and "WZW action" plus some couplings which allow the formula to globalize]

For $H \neq 1$, we'll do similarly. But now construction of ω_o is trickier.

Use projection $\mathfrak{g} \to \mathfrak{h}$ on ω to get conn. ω_H on $P \to Q$.

$$\begin{tikzcd}
& \pi^*P \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"', "{\pi_1^*\omega,\ \omega_o}"] & \\
P \arrow[rr, "\omega_H"] \arrow[dr, "\pi_3"', "\omega"] & & Q \arrow[dl, "\pi"] \\
& M &
\end{tikzcd}$$

Then $P\times_H G = \pi^*P$ (left factor is $P \to Q$)

$$(p, g) \longmapsto (p, (g, 1))$$

[over $M = pt$, this would say $G\times_H G = G \times G/H$]

so ω_H induces a conn ω_o on $\pi^*P \longrightarrow Q$.

Look at $\pi^*P\times I \longrightarrow Q\times I$ with conn

$$\underline{\omega} = (1-t)\omega_o + t\,\pi_1^*\omega$$

Then once again $d\left[\int_I \Phi(\underline{\omega})\right] = \Phi(\pi_1^*\Omega) - \cancelto{0}{\Phi(\Omega_o)}$ and $\int_I \Phi(\underline{\omega}) = \pi_3^*\beta$ as before.