Killing fields

Say M Riemannian, G acts on M by isometries.

Let $Y = \sigma(X)$ $\quad X \in \mathfrak{g}$

Prop Y obeys $\mathcal{L}_Y g = 0$.

Pf Take $h : (-\varepsilon, \varepsilon) \to G$ $\quad h(0) = 1$ $\quad h'(0) = X$

Then $h(t)^* g = g$ so $\displaystyle 0 = \lim_{t\to 0} \frac{h(t)^* g - g}{t} = \lim_{t \to 0} \frac{\exp_{tY}^* g - g}{t} = \mathcal{L}_Y g.$

Def K is a Killing vector field if $\mathcal{L}_K g = 0$.

Prop K Killing $\Rightarrow \langle \nabla_X K, Z\rangle + \langle \nabla_Z K, X\rangle = 0 \quad \forall\, X, Z \in TM$

Pf $K(\langle X, Z\rangle) = (\mathcal{L}_K g)(X,Z) + \langle \mathcal{L}_K X, Z\rangle + \langle X, \mathcal{L}_K Z\rangle$

$$\langle \nabla_K X, Z\rangle + \langle X, \nabla_K Z\rangle = \langle [K,X], Z\rangle + \langle X, [K,Z]\rangle$$

$$\langle \nabla_X K, Z\rangle + \langle X, \nabla_Z K\rangle = 0$$

Prop Y Killing, γ geodesic in M $\Rightarrow \dfrac{d}{dt}\langle Y, \dot\gamma(t)\rangle = 0$

Pf $$\frac{d}{dt}\langle Y, \dot\gamma(t)\rangle = \underbrace{\cancel{\langle \nabla_{\dot\gamma} Y, \dot\gamma\rangle}}_{0 \text{ by Killing eq.}} + \underbrace{\cancel{\langle Y, \nabla_{\dot\gamma}\dot\gamma\rangle}}_{0 \text{ by geodesic eq.}}$$

Thus "an isometry leads to a conserved quantity." (A special case of Noether's Thm)

Useful for describing the geodesics. e.g. on S^2, $\frac{\partial}{\partial\phi}$ is Killing, so we get immediately

$$\frac{d}{dt}\langle \dot\gamma, \partial/\partial\phi\rangle = 0.$$

$g\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi}\right) = \sin^2\theta$ so this says $\dot\phi \sin^2\theta = \text{const}.$