Lorentzian geometry

Def A Lorentzian metric on M is $g \in \mathcal{E}(\mathrm{Sym}^2(T^*M))$ which is everywhere of signature $(\underbrace{+1,\dots,+1}_{n-1}, -1)$.

In general relativity, spacetime is a 4-manifold M w/ Lorentzian metric.

Essentially all local formulas which we wrote in this course continue to make sense in Lorentzian setting.

e.g. geodesic equation. "Test particles" follow geodesics in M.

Ex $M = \mathbb{R}^{3,1}$. $g = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$

Geodesics: $\gamma(\tau) = p + v \cdot \tau$ $\quad p \in \mathbb{R}^{3,1}, v \in \mathbb{R}^{3,1}$

Assuming $\|v\|^2 < 0$ (timelike)

we may rescale τ so that $\|v\|^2 = -1$. Then call τ proper time.

(It is the time as experienced "by the particle.")

$$v = \begin{pmatrix} \sqrt{1+\vec{v}^2} \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Physics invariant under $\mathrm{Isom}(M) = \mathbb{R}^{3,1} \rtimes O(3,1)$

"Lorentz transformations"

The metric in M is constrained by Einstein equation:

$$\mathrm{Ric} - \tfrac{1}{2} g \cdot S = T$$

where T is the energy-momentum tensor. (Obeys $\nabla_\mu T^{\mu\nu} = 0$)

In particular, in vacuum, $T = 0$, so Einstein eq says

$$\mathrm{Ric} - \tfrac{1}{2} g \cdot S = 0$$

ie $\mathrm{Ric}_{ij} - \frac{1}{2} g_{ij} S = 0$

Taking traces, $\mathrm{Ric}_i{}^i - \frac{1}{2} g_i{}^i S = 0$

$$S - 2S = 0 \implies S = 0$$

Thus $\mathrm{Ric} = 0$ (Vacuum Einstein equation)

Ex (Schwarzschild)

What is the metric created by a spherically symmetric, static distribution of matter?

$$g = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\left[d\theta^2 + \sin^2\theta\, d\varphi^2\right] = \begin{pmatrix} -A & & & \\ & B & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix}$$

Now "just" compute:

e.g. $\Gamma^r_{rr} = \frac{1}{2}B^{-1}B'$, etc.

Find: $Ric_{rr} + Ric_{tt} = 0 \Rightarrow A'B + AB' = 0$

$Ric_{\varphi\varphi} = 0 \Rightarrow r(A'B - B'A) + 2AB(A-B) = 0$

First eq. says $AB = K$; by rescaling t we may arrange $K = 1$ so $A = B^{-1}$

Then second eq. simplifies to $rB' + B = 1$ ie $\frac{d}{dr}(rB) = 1$

which has the solutions $B(r) = \left(1 - \frac{C}{r}\right)$ for any C.

Geodesics

Suppose we consider motion with $\theta = \frac{\pi}{2}$ (equatorial plane).

Killing vectors: $\partial/\partial t$, $\partial/\partial\varphi$

Thus $Q_1 = \left(1 - \frac{C}{r}\right)\dot{t}$, $Q_2 = r^2\dot{\varphi}$ are conserved.

Then we have $-\left(1-\frac{C}{r}\right)\dot{t}^2 + \left(1-\frac{C}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 = -1$ for timelike geodesics

$$-\left(1-\frac{C}{r}\right)^{-1}Q_1^2 + \left(1-\frac{C}{r}\right)^{-1}\dot{r}^2 + \frac{1}{r^2}Q_2^2 = -1$$

$$-Q_1^2 + \dot{r}^2 + \frac{1}{r^2}Q_2^2\left(1-\frac{C}{r}\right) = -\left(1-\frac{C}{r}\right)$$

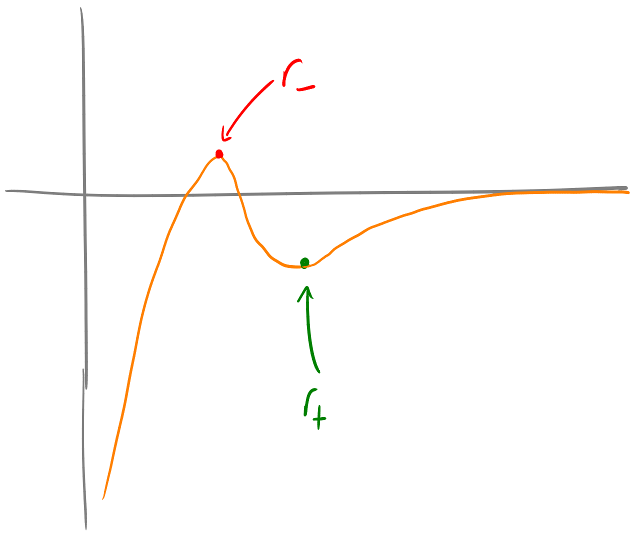
$$\dot{r}^2 = -\left(1-\frac{C}{r}\right) + Q_1^2 - \frac{Q_2^2}{r^2} + \frac{C\,Q_2^2}{r^3}$$

$$\ddot{r} = -\frac{C}{2r^2} + \frac{Q_2}{r^3} - \frac{3}{2}C\frac{Q_2^2}{r^4} = -\frac{\partial V}{\partial r}$$

finally set $C = 2M$, $Q_2 = \ell$

$$V(r) = \underbrace{-\frac{M}{r} + \frac{\ell^2}{2r^2}}_{\text{Newtonian physics}} \underbrace{- \frac{M\cdot\ell^2}{r^3}}_{\text{"relativistic correction"}}$$

For $\ell^2 > 12M^2$, $V(r)$ looks like:



$$r_\pm = \frac{\ell^2 \pm \sqrt{\ell^4 - 12\ell^2 M^2}}{2M}$$

$r_+ > 6M \Rightarrow$ no stable circular orbits at $r \leq 6M$

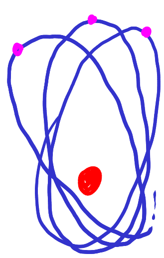
Small oscillations around r_+ have frequencies $\omega_r^2 = \left.\frac{d^2V}{dr^2}\right|_{r_+} = \frac{M(r_+ - 6M)}{r_+^3(r_+ - 3M)}$

while the angular frequency is $\omega_\varphi^2 = \dot{\varphi}^2 = \frac{\ell^2}{r_+^4} = \frac{M}{r_+^2(r_+ - 3M)}$

The difference between these gives the precession: $\omega_p = \omega_\varphi - \omega_r = \left(1 - \sqrt{1 - 6M/r_+}\right)\omega_\varphi$

(for nearly circular orbits)

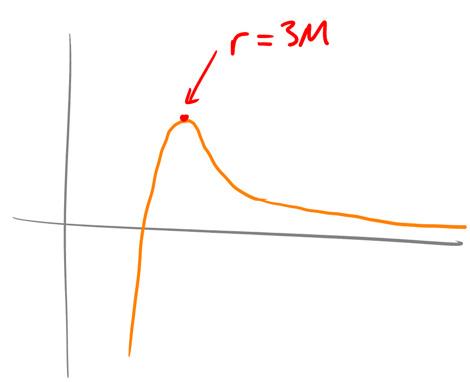
$$\approx \frac{3M^{3/2}}{r_+^{5/2}} \quad \text{for } r_+ \gg M$$



Rk: in our units, $M_{Earth} \approx 4.4$ mm, $M_{Sun} \approx 1.5$ km

For Mercury, $r \approx 58$ million km
$\omega_p \approx .00021$ rad/century [observed!]

For lightlike geodesics, get $V(r) = \frac{\ell^2}{2r^3}(r - 2M)$



So, have an unstable circular orbit at $r = 3M$.

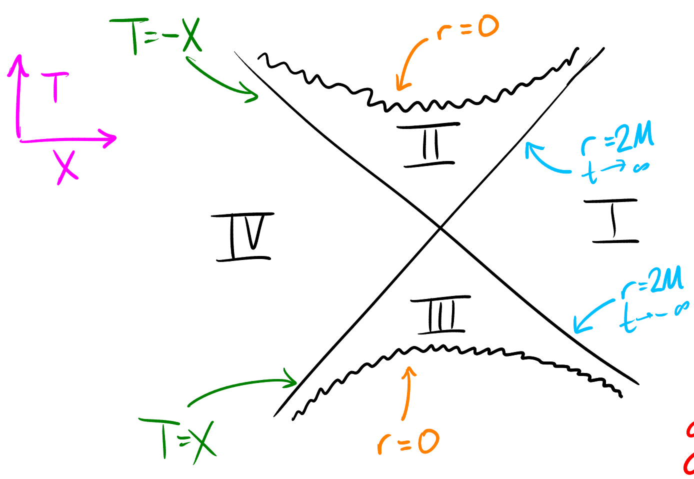
The metric g as written so far is defined on $M = \{(t,r,\theta,\varphi) \in \mathbb{R}^2 \times S^2 : r > 2M\}$

Geodesically <u>incomplete</u>: geodesics reach $r = 2M$ in finite τ.

The metric appears to be singular at $r = 2M$. But, no <u>invariant</u> is blowing up in that limit.

And indeed, there is a <u>geodesically complete</u> Ricci-flat Lorentzian $\tilde{M} \supset M$.

$$\tilde{M} = \{(T, X, \theta, \varphi) \in \mathbb{R}^2 \times S^2 : T^2 - X^2 < 1\}.$$



$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = X^2 - T^2$$

$$\frac{t}{2M} = 2\tanh^{-1}\left(T/X\right)$$

$$g = \frac{32M^3 e^{-r/2M}}{r}\left(-dT^2 + dX^2\right) + r^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

Any timelike geodesic in region II will reach the singularity at $T^2 - X^2 = 1$, in finite τ.

Thus region II is "black hole" $r = 2M$ is "event horizon".

<u>Thm</u> (Birkhoff) Every spherically symmetric Ricci-flat spacetime is $\subset \tilde{M}$.