Lecture 10

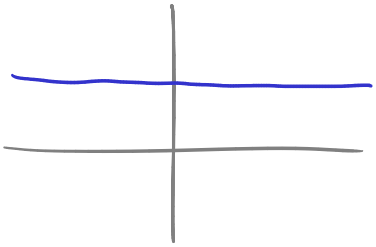
21 Sep 2018

Exam 1 Oct 1 (week from Monday)

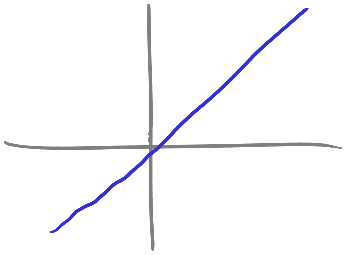
- only need pencils + erasers, ID
- no calculators
- problems very similar to Quest HW ≈ 12-13 of them
- covers all material from any lecture before exam day
- during class time, in usual room

Last time: derivatives $\frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Recall: $\frac{d}{dx}(c) = 0$



$\frac{d}{dx}(x) = 1$



Fact $\frac{d}{dx}(x^n) = nx^{n-1}$

Why? If $f(x) = x^n$

$$f'(a) = \lim_{x \to a} \frac{x^n - a^n}{x-a}$$

And $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})$

(eg $x^4 - a^4 = (x-a)(x^3 + x^2a + xa^2 + a^3)$)

So $\lim_{x \to a} \frac{x^n - a^n}{x-a} = \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + a^{n-1})}{x-a}$

$= \lim_{x \to a} x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + a^{n-1}$

$$= \underbrace{a^{n-1} + a^{n-1} + a^{n-1} + \cdots + a^{n-1}}_{n \text{ terms}}$$

$$= n \cdot a^{n-1}$$

Ex $\frac{d}{dx}(x^4) = 4x^3 \qquad \frac{d}{dt}(t^9) = 9t^8$

Actually this works for any exponent:

Power rule $\frac{d}{dx}(x^r) = rx^{r-1}$ for r any constant $r \neq 0$

Ex $\frac{d}{dx}\sqrt{x} = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d}{dx}(x^{-3}) = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$

Constant Multiple Rule $\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}f(x)$

[Why? $\lim_{h\to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} = c \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = c \cdot \frac{d}{dx}f(x)$]

Ex $\frac{d}{dx}\left(-\frac{7}{x^2}\right) = -7\frac{d}{dx}\left(\frac{1}{x^2}\right) = -7 \cdot \frac{d}{dx}(x^{-2}) = -7 \cdot (-2x^{-2-1})$

$= 14x^{-3} = \frac{14}{x^3}$

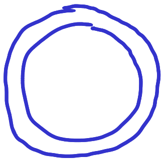
Ex $\frac{d}{dr}(\pi r^2) = \pi \cdot \frac{d}{dr}(r^2) = \pi \cdot 2r^{2-1} = 2\pi r$

area of a circle

circumference of a circle

[Rk People sometimes write this as

$d(\pi r^2) = 2\pi r\, dr$ instead of $\frac{d}{dr}(\pi r^2) = 2\pi r$]



Sum Rule $\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$ (if both f and g are diff'ble at x)

(exercise: prove this!)

Ex

$$\frac{d}{dx}\left(\frac{3-5x^3}{\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{3}{\sqrt{x}} + \frac{-5x^3}{\sqrt{x}}\right)$$

$$= 3\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) + -5\frac{d}{dx}\left(\frac{x^3}{\sqrt{x}}\right)$$

$$= 3\frac{d}{dx}\left(x^{-1/2}\right) + -5\frac{d}{dx}\left(x^{5/2}\right) \quad \leftarrow x^3 \cdot x^{-1/2}$$

$$= 3\cdot\left(-\tfrac{1}{2}\right)x^{-3/2} + -5\cdot\left(\tfrac{5}{2}\right)x^{3/2}$$

$$= \underline{\underline{-\tfrac{3}{2}x^{-3/2} - \tfrac{25}{2}x^{3/2}}}$$

Exponential functions

Fact There is a number, e, such that $\lim\limits_{h\to 0}\frac{e^h-1}{h} = 1$.

$(e \approx 2.71828\ldots)$

Fact $\frac{d}{dx}\left(e^x\right) = e^x$ (not xe^{x-1})

$\underset{\uparrow\,\text{const}}{e}^{\overset{\text{var.}}{x}}$ is not $\underset{\uparrow\,\text{var.}}{x}^{\overset{\text{const}}{r}}$

Why?

$$\lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h}$$

$$= \lim_{h\to 0} e^x\left(\frac{e^h-1}{h}\right)$$

$$= e^x \lim_{h\to 0}\frac{e^h-1}{h} = e^x\cdot 1 = e^x$$

Ex $\frac{d}{dx}\left(x+7e^x\right) = 1+7e^x$

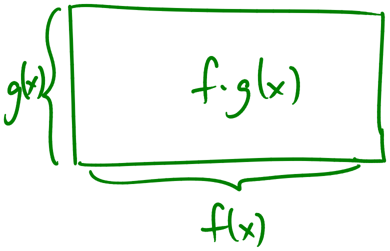
Product Rule $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)\,g(x) + f(x)\,g'(x)$ (NOT $f'(x)g'(x)$)

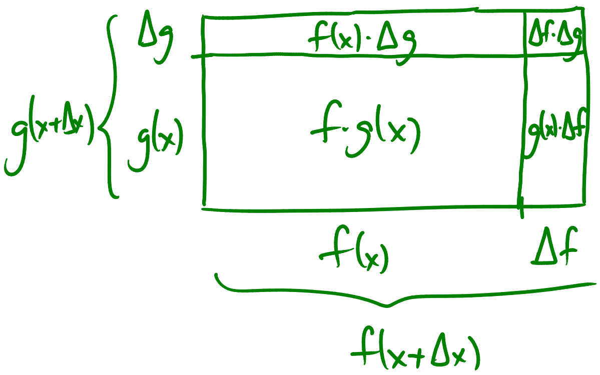
if both f, g diff'ble at x

Ex $$\frac{d}{dx}\left(\underbrace{x^3}_{f(x)}\underbrace{e^x}_{g(x)}\right) = \underbrace{3x^2}_{f'(x)}\underbrace{e^x}_{g(x)} + \underbrace{x^3}_{f(x)}\underbrace{e^x}_{g'(x)} = \underline{\underline{e^x(3x^2+x^3)}}$$

Why does it work? Want to calculate $\lim_{\Delta x\to 0}\frac{\Delta(fg)}{\Delta x}$

$f \times g =$





ie $\Delta(fg) = f(x)\,\Delta g + g(x)\,\Delta f + \Delta f\,\Delta g$

$$\lim_{\Delta x\to 0}\frac{\Delta(fg)}{\Delta x} = \lim_{\Delta x\to 0} f(x)\frac{\Delta g}{\Delta x} + g(x)\frac{\Delta f}{\Delta x} + \frac{\Delta f\cdot\Delta g}{\Delta x}$$

$$= f(x)\,g'(x) + g(x)\,f'(x) + 0$$ as we wanted!