Lecture 25 | 29 Oct 2018

Midterm 2 Friday Nov 2
covers HW5 – HW9

HW10 not on this exam
will be due next Wed morning (1-day extension)

You need not memorize any trig ident. except $\sin^2 x + \cos^2 x = 1$

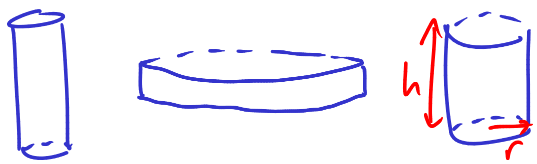
$\tan^2 x + 1 = \sec^2 x$

My office hrs: M 2–3
W 3:30 – 4:30
Th 10:30 – 11:30 ← extra

RLM 9.134

Optimization

Ex A cylindrical vat without a top is to hold V cm³ of liquid. What are the dimensions for the vat which minimize the surface area?



$$V = \underbrace{\pi r^2}_{\text{area of base}} \cdot h \leftarrow \text{height}$$

Want to minimize surface area:

$$A = \underbrace{\pi r^2}_{\substack{\text{area of}\\ \text{bottom}}} + \overbrace{2\pi r \cdot h}^{\text{area of side}}$$

$2\pi r$ ← circumf. of base

Eliminate h using our constraint:

$$V = \pi r^2 h$$

$$\frac{V}{\pi r^2} = h$$

Then $A = \pi r^2 + 2\pi r \cdot \frac{V}{\pi r^2}$

$= \pi r^2 + 2\frac{V}{r}$

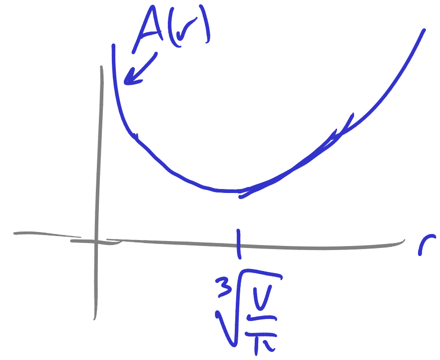
Function of one variable r, domain $(0, \infty)$

Look for critical pts: $\frac{dA}{dr} = 2\pi r - 2\frac{V}{r^2} = 0$

$$\longrightarrow \quad 2\pi r = 2\frac{V}{r^2}$$

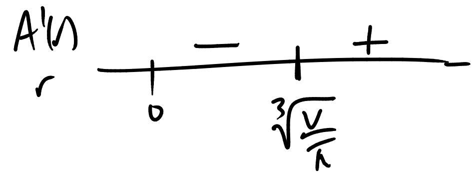
$$r^3 = \frac{V}{\pi}$$

$$r = \sqrt[3]{\frac{V}{\pi}}$$



(check: $\frac{dA}{dr} = 2\pi r - 2\frac{V}{r^2}$

$= 2\pi r\left(1 - \frac{V}{\pi r^3}\right)$)



$\Longrightarrow$ local min ✓

So, the absolute minimum is attained at

$$r = \sqrt[3]{\frac{V}{\pi}}.$$

$$h = \frac{V}{\pi r^2} = \frac{V}{\pi\left(\frac{V}{\pi}\right)^{2/3}} = \left(\frac{V}{\pi}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{V}{\pi}}$$

Alternate method for finding the critical pt:

$$V = \pi r^2 h \qquad A = \pi r^2 + 2\pi r h$$

Instead of eliminating a variable, can use implicit differentiation: apply $\frac{d}{dr}$ to both equations

$$0 = \frac{dV}{dr} = 2\pi r h + \pi r^2 \frac{dh}{dr} \qquad 0 = \frac{dA}{dr} = 2\pi r + 2\pi h + 2\pi r\frac{dh}{dr}$$

V is constant — at critical point

$$2\pi r h + \pi r^2 \frac{dh}{dr} = 0 \qquad 2\pi r + 2\pi h + 2\pi r \frac{dh}{dr} = 0$$

$$2rh + r^2\frac{dh}{dr} = 0 \qquad r + h + r\frac{dh}{dr} = 0$$

$$\frac{dh}{dr} = -\frac{2h}{r} \qquad r + h + r\left(-\frac{2h}{r}\right) = 0$$

$$r + h - 2h = 0$$

$$\underline{\underline{r = h}}$$

Once we know $r = h$, use $V = \pi r^2 h$

$$\longrightarrow V = \pi r^3 \quad \text{so} \quad r = \sqrt[3]{\frac{V}{\pi}}$$

Newton's method

Suppose we want to solve the equation $f(x) = 0$
where $f(x) = x^3 - 2x - 5$.

(Galois: there's no analog of quadratic formula for eq. of degree ≥ 6.)

How do we do it?

x	$f(x)$
1	-6
2	-1
3	16

IVT $\Rightarrow$ there is a solution $x \in (2,3)$. But what is this x?

To find it: start with an initial approximate solution x_1 (here we could take $x_1 = 2$)

Then, get a new approximate solution by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

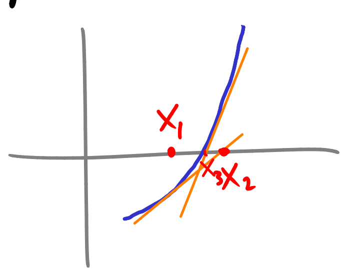
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$\vdots$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Why should this work?



We write the linear approx to $f(x)$ at x_1:

$$f(x) \approx f(x_1) + f'(x_1)(x - x_1)$$

set this equal to 0

then $0 = f(x_1) + f'(x_1)(x - x_1)$

$$-\frac{f(x_1)}{f'(x_1)} = x - x_1$$

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

then call this x_2

In our case $f(x) = x^3 - 2x - 5$

$f'(x) = 3x^2 - 2$

Here taking $x_1 = 2$ $\qquad f(2) = -1$

$f'(2) = 10$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2 - \frac{-1}{10} = 2 + \frac{1}{10} = 2.1$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.1 - \frac{f(2.1)}{f'(2.1)} \approx 2.0946$$

already correct to 4 decimal places!