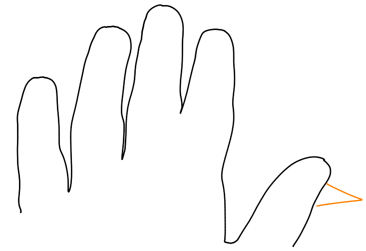
Lecture 33 19 Nov 2018



Last time: substitution rule for indef $\int$

Ex $\int \sqrt{1+x^2}\, x^5\, dx$

$u = 1+x^2$

$du = 2x\, dx$

$u-1 = x^2$

$(u-1)^2 = x^4$

$$= \int \sqrt{u} \cdot \frac{x^4}{2}\, 2x\, dx$$

$$= \int \sqrt{u} \cdot \frac{x^4}{2}\, du$$

$$= \int \sqrt{u} \cdot \frac{(u-1)^2}{2}\, du$$

$$= \frac{1}{2}\int \sqrt{u}\cdot (u-1)^2\, du$$

$$= \frac{1}{2}\int \sqrt{u}\cdot (u^2-2u+1)\, du$$

$$= \frac{1}{2}\int u^{5/2} - 2u^{3/2} + u^{1/2}\, du$$

$$= \cdots$$

or: $\int \sqrt{u}\cdot x^5\, dx$

$du = 2x\, dx$

$\frac{du}{2x} = dx$

$\int = \int \sqrt{u}\cdot x^5 \cdot \frac{du}{2x}$

$= \int \sqrt{u}\cdot \frac{x^4}{2}\, du$

Substitution for definite integrals: very similar to indefinite, but have to remember to transform the endpoints too!

Ex $\int_0^{\pi/2} \sin(2x)\, dx$

$u = 2x$

$du = 2\, dx$

$\frac{1}{2} du = dx$

$$= \int_{x=0}^{x=\pi/2} \sin(2x)\, dx$$

$$= \int_{u=0}^{u=\pi} \sin(u)\cdot \frac{1}{2} du$$

$$= \frac{1}{2}\int_0^{\pi} \sin(u)\, du = \frac{1}{2}\left(-\cos u \Big|_0^{\pi}\right)$$

$$= \frac{1}{2}\left((-\cos\pi) - (-\cos 0)\right)$$

$$= \frac{1}{2}\left(-(-1) - (-1)\right) = \frac{1}{2}(1+1) = 1$$

Ex $\displaystyle\int_{\pi/3}^{\pi/2} (\cos 3x)\, \underbrace{e^{(\sin 3x)}}_{e^u}\, dx$

$\frac{1}{3}du$

$u = \sin 3x$

$du = 3\cos 3x\, dx \longrightarrow \frac{1}{3}\, du = \cos 3x\, dx$

$x = \pi/3 \longrightarrow u = \sin(3 \cdot \pi/3) = 0$

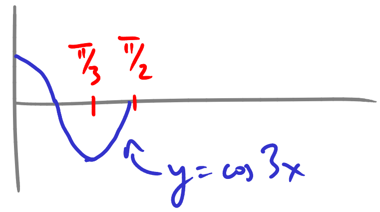
$x = \pi/2 \longrightarrow u = \sin(3 \cdot \pi/2) = -1$

$$= \int_0^{-1} e^u \cdot \frac{1}{3}\, du$$

$$\left(= -\int_{-1}^{0} e^u \cdot \frac{1}{3}\, du \right)$$

$$= \frac{1}{3} \cdot e^u \Big|_0^{-1}$$

$$= \frac{1}{3}\left(e^{-1} - e^0\right) = \underline{\frac{1}{3}\left(\frac{1}{e} - 1\right)}$$



Remark: instead of transforming the endpts, can also put the indef $\int$ back in terms of x and then use the original endpoints.

e.g.

$$\int_{x=\pi/3}^{x=\pi/2} e^u \cdot \frac{1}{3}\, du = \frac{1}{3}\left(e^u \Big|_{x=\pi/3}^{x=\pi/2} \right)$$

$$= \frac{1}{3}\left(e^{\sin 3x} \Big|_{x=\pi/3}^{x=\pi/2} \right)$$

$$= \underline{\frac{1}{3}\left(e^{-1} - 1\right)}$$

Ex $\displaystyle\int \tan x\, dx$

$u = \cos x$

$du = -\sin x\, dx$

$-du = \sin x\, dx$

$$= \int \frac{\sin x}{\cos x}\, dx$$

$$= \int \frac{-du}{u}$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

$$= \ln|\cos x|^{-1} + C = \ln|\sec x| + C$$

$$\int \frac{dx}{\sqrt{x}(1+x)} \Longrightarrow \text{try } u = 1+x: \quad du = dx, \quad x = u-1 \longrightarrow \int \frac{du}{\sqrt{u-1} \cdot u}$$

not helping

$\Downarrow$

try $u = \sqrt{x}$:

$$du = \frac{1}{2\sqrt{x}}\,dx$$

$$2\,du = \frac{dx}{\sqrt{x}}$$

so $\displaystyle\int \frac{dx}{\sqrt{x}(1+x)} = \int \frac{2\,du}{1+x}$

and $u^2 = x$, so $\displaystyle = \int \frac{2\,du}{1+u^2} = 2\tan^{-1}(u) + C$

$$= 2\tan^{-1}(\sqrt{x}) + C$$

Ex $\displaystyle\int \underbrace{\tan^2\theta}_{u^2}\ \underbrace{\sec^2\theta\, d\theta}_{du}$

$u = \tan\theta$

$du = \sec^2\theta\, d\theta$

[$u = \sec\theta$ $\quad u = \sec\theta\tan\theta$

all give more complicated $\int$ . .]

$$= \int u^2\, du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3\theta + C$$

Ex $\displaystyle\int \frac{dx}{1+9x^2}$

$u = 3x \longrightarrow u^2 = 9x^2$

$du = 3\,dx$

$\frac{1}{3}du = dx$

$$= \int \frac{\frac{1}{3}\,du}{1+u^2}$$

$$= \frac{1}{3}\tan^{-1}(u) + C = \frac{1}{3}\tan^{-1}(3x) + C$$

Ex $\displaystyle\int \frac{5}{x^2+6x+10}\,dx$

Want to make this look like $\dfrac{(\text{something})}{u^2+1}$

The trick: "completing the square" — set $u = x+3$

then $u^2 = x^2+6x+9$ $\quad du = dx$

so $\int \frac{5}{x^2+6x+10}\,dx = \int \frac{5}{u^2+1}\,du$

(in general: for denom x^2+Ax+B try $u = x+\frac{1}{2}A$)

$$= 5\tan^{-1}u + C$$
$$= \underline{5\tan^{-1}(x+3) + C}$$

Ex $\int \frac{dx}{x^2+6x+11}$ $\quad u = x+3$

$$= \int \frac{du}{u^2+2}$$

$w = \frac{u}{\sqrt{2}}$ $\quad u = w\cdot\sqrt{2}$ $\quad u^2 = 2w^2$

$dw = \frac{du}{\sqrt{2}}$ $\quad du = \sqrt{2}\,dw$

$$= \int \frac{\sqrt{2}\,dw}{2w^2+2}$$

$$= \frac{\sqrt{2}}{2}\int \frac{dw}{w^2+1} = \frac{1}{\sqrt{2}}\tan^{-1}(w) = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) = \underline{\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right)}$$