Lecture 10

24 Sep 2012

Housekeeping: First midterm exam Oct 9 7pm

Last time: trigonometric integrals

Like $\int \sin^a\theta \cos^b\theta \, d\theta$

or $\int \sec^a\theta \tan^b\theta \, d\theta$

See web site for $I = \int \sec^3 x \, dx$

Trigonometric Substitution (Ch 7.3)

Ex $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \ ?$

$[-3 \le x \le 3]$

$\left[-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right]$

A clever substitution: $x = 3\sin\theta$

$dx = 3\cos\theta \, d\theta$

$$\sqrt{9-x^2} = \sqrt{9-(3\sin\theta)^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{1-\sin^2\theta}$$

$$= 3\sqrt{\cos^2\theta} = 3\cos\theta$$ [because $\cos\theta > 0$]

So
$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{(3\sin\theta)^2}\cdot 3\cos\theta\,d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cot^2\theta\,d\theta$$

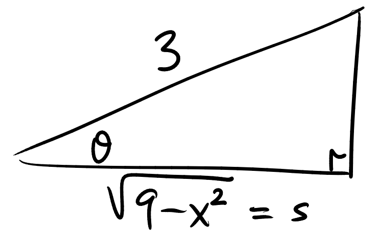
$$= \int (\csc^2\theta - 1)\,d\theta$$

$$= -\cot\theta - \theta + C$$

To re-express this in terms of x: $\qquad x = 3 \sin\theta$

$$\frac{x}{3} = \sin\theta \rightsquigarrow \theta = \sin^{-1}\left(\frac{x}{3}\right)$$

$$\frac{x}{3} = \sin\theta = \frac{\text{opp}}{\text{hyp}}$$



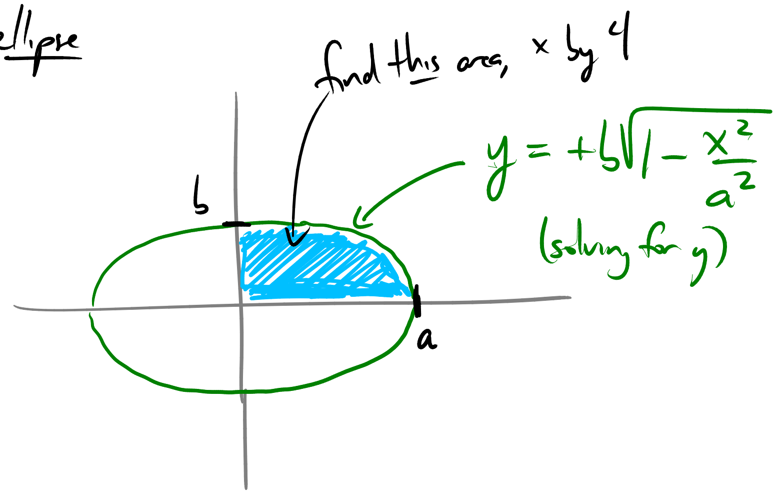
$$s^2 + x^2 = 3^2$$

$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{9-x^2}}{x}$$

So finally the integral is: $\qquad \underline{-\dfrac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\dfrac{x}{3}\right) + C}$

Ex Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$A = 4\int_0^a dx\, b\sqrt{1 - \frac{x^2}{a^2}}$$

$$= 4b\int_0^a dx\sqrt{1 - \frac{x^2}{a^2}}$$

[Want this to become $\sqrt{1-\sin^2\theta}$ $\qquad$ So, put $\dfrac{x^2}{a^2} = \sin^2\theta$

$x^2 = a^2\sin^2\theta$

$x = a\sin\theta$]

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$x = 0 \rightsquigarrow a\sin\theta = 0 \longrightarrow \theta = 0$

$x = a \rightsquigarrow a\sin\theta = a \rightsquigarrow \theta = \pi/2$

$$\int = 4b\int_0^{\pi/2} a\cos\theta\, d\theta\,\sqrt{1-\sin^2\theta}$$

$$= 4ab \int_0^{\pi/2} d\theta \, \cos\theta \cdot \sqrt{\cos^2\theta}$$

$$= 4ab \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= 4ab \int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\, d\theta \qquad [\tfrac{1}{2}\text{-angle id}]$$

$$= \cdots$$

$$= \underline{\underline{ab\pi}}$$

<u>Ex</u> $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\, dx$

Here use the identity $\tan^2 + 1 = \sec^2$:

Substitute $x = 2\tan\theta$, $dx = 2\sec^2\theta \, d\theta$

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = 2\sqrt{\tan^2\theta + 1} = 2\sqrt{\sec^2\theta} = 2\sec\theta$$

$$\longrightarrow \int = \int \frac{1}{4\tan^2\theta \cdot 2\sec\theta} \cdot 2\sec^2\theta \, d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{\cos^2\theta}{\sin^2\theta} \cdot \frac{1}{\cos\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta \qquad u = \sin\theta, \quad du = \cos\theta \, d\theta$$

$$= \frac{1}{4}\int \frac{du}{u^2}$$

$$= -\frac{1}{4}\left(\frac{1}{u}\right) + C$$

Go back to original variable:

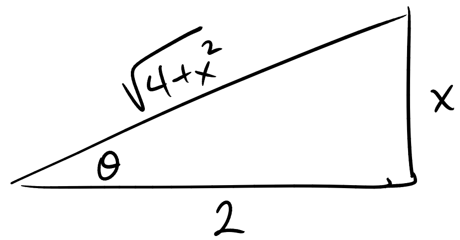
$u = \sin\theta$

$x = 2\tan\theta$

$\frac{x}{2} = \tan\theta$

What's u in terms of x?

$$u = \sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{4+x^2}}$$



So finally, $$\int = \underline{\underline{-\frac{1}{4}\left(\frac{\sqrt{4+x^2}}{x}\right) + C}}$$

How to pick a trig. substitution?

For $\sqrt{a^2 - x^2}$ use $x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\quad 1 - \sin^2\theta = \cos^2\theta$

$\sqrt{a^2 + x^2}$ use $x = a\tan\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\quad 1 + \tan^2\theta = \sec^2\theta$

$\sqrt{x^2 - a^2}$ use $x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$ $\quad \sec^2\theta - 1 = \tan^2\theta$

Ex $$\int \frac{dx}{\sqrt{x^2 + 8x + 25}}$$

Want to relate this to s.t. like $$\int \frac{}{\sqrt{u^2 + a^2}}$$

Completing the square: try $u = x + c$ for some const. c

$$u^2 = x^2 + 2c \cdot x + c^2$$

so $$u^2 + a^2 = x^2 + 2c\cdot x + c^2 + a^2$$

Want this to $$= x^2 + 8x + 25$$

$\Longrightarrow$ we should take $\underline{c = 4}$

i.e. $u = x+4$ $\quad x = u-4 \quad dx = du$

Then $u^2 = (x+4)^2 = x^2+8x+16$

$\longrightarrow \quad x^2+8x+25 = u^2+9$

So, $$\int \frac{dx}{\sqrt{x^2+8x+25}} = \int \frac{du}{\sqrt{u^2+9}}$$

Substitute $u = 3\tan\theta \qquad du = 3\sec^2\theta \, d\theta$

So $$\int = \int \frac{3\sec^2\theta \, d\theta}{\sqrt{9+9\tan^2\theta}}$$

$$= \int \frac{3\sec^2\theta \, d\theta}{3\sqrt{\sec^2\theta}}$$

$$= \int \frac{3\sec^2\theta}{3\sec\theta} \, d\theta = \int \sec\theta \, d\theta = \ln|\sec\theta+\tan\theta| + C$$

Write it in terms of x: $\quad u = 3\tan\theta \longrightarrow \tan\theta = \frac{u}{3} = \frac{x+4}{3}$

To get $\sec\theta$ in terms of x:

could draw $\triangle$, use SOHCAHTOA.