Lecture 11

26 Sep 2012

Housekeeping: HW06 due Tue Oct 2
HW07 due Tue Oct 9 ⟵ !
EXAM 1 Tue Oct 9

Last time: Trigonometric substitution
e.g. for $\int$ involving $\sqrt{1-x^2}$, substitute $x = \sin\theta$
use identity $1 - \sin^2\theta = \cos^2\theta$
then $\sqrt{1-x^2} = \sqrt{\cos^2\theta} = \cos\theta \ldots$

Partial fractions (Sec 7.4)

How to integrate complicated rational functions $\dfrac{P(x)}{Q(x)}$

P, Q polynomials

Ex
$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$$

Factor the denominator: $2x^3+3x^2-2x = x(2x^2+3x-2)$
$= x(2x-1)(x+2)$

Then set
$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

To find A, B, C: mult. both sides by the denom. $x(2x-1)(x+2)$

$$x^2+2x-1 = A\cdot(2x-1)(x+2) + B\cdot x(x+2) + C\cdot x(2x-1)$$
$$x^2+2x-1 = A\cdot(2x^2+3x-2) + B(x^2+2x) + C\cdot(2x^2-x)$$

$$1\cdot x^2+2\cdot x+-1 = (2A+B+2C)x^2+(3A+2B-C)x+(-2A)$$

Equate the coefficients:

$$\left.\begin{aligned} 1 &= 2A+B+2C \\ 2 &= 3A+2B-C \\ -1 &= -2A \end{aligned}\right\} \rightarrow A=\tfrac{1}{2} \quad \begin{aligned} 1 &= 1+B+2C \\ 2 &= \tfrac{3}{2}+2B-C \end{aligned}$$

Solve these equations: $\rightarrow A=\frac{1}{2},\ B=\frac{1}{5},\ C=-\frac{1}{10}$

So, $$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{1}{2}\cdot\frac{1}{x}+\frac{1}{5}\cdot\frac{1}{2x-1}-\frac{1}{10}\cdot\frac{1}{x+2}$$

$$\int\frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \int\frac{1}{2}\cdot\frac{1}{x}+\frac{1}{5}\frac{1}{2x-1}-\frac{1}{10}\frac{1}{x+2}$$

$$= \underline{\underline{\frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|-\frac{1}{10}\ln|x+2|+K}}$$

What if the denom. doesn't factor completely into linear factors?

Ex $\displaystyle\int\frac{2x^2-x+4}{x^3+4x}\,dx$ Factor: $x^3+4x = x(x^2+4)$

Write $$\frac{2x^2-x+4}{x^3+4x} = \frac{A}{x}+\frac{Bx+C}{x^2+4}$$

To find A, B, C: $\times$ both sides by $x(x^2+4)$

$$\begin{aligned} 2x^2-x+4 &= A(x^2+4)+(Bx+C)x \\ &= Ax^2+4A+Bx^2+Cx \\ &= (A+B)x^2+(C)x+4A \end{aligned}$$

$$\left.\begin{aligned} 2 &= A + B \\ -1 &= C \\ 4 &= 4A \end{aligned}\right\} \longrightarrow \begin{aligned} A &= 1 \\ B &= 1 \\ C &= -1 \end{aligned}$$

So, $\int = \int \frac{1}{x} + \frac{x-1}{x^2+4}\,dx$

$$= \int \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\,dx$$

$\ln|x|$ — use $u = x^2+4$ get $\frac{1}{2}\ln(x^2+4)$ — use $u = \frac{x}{2}$ get $-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$

$$= \underline{\underline{\ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + K}}$$

What if the degree of the numerator $\geq$ the degree of the denominator?

Ex $\int_0^1 \frac{x^3 - 4x - 10}{x^2 - x - 6}\,dx$

Divide first:

$$\begin{array}{r} x + 1 \\ x^2 - x - 6 \overline{)\, x^3 + 0x^2 - 4x - 10} \\ \underline{x^3 - x^2 - 6x} \\ x^2 + 2x - 10 \\ \underline{x^2 - x - 6} \\ 3x - 4 \end{array}$$

So

$$\int \frac{x^3 - 4x - 10}{x^2 - x - 6}\,dx = \int x + 1 + \underbrace{\frac{3x-4}{x^2-x-6}}\,dx$$

still need to do partial fractions on this part!

Factor: $x^2 - x - 6 = (x-3)(x+2)$

$$\frac{3x-4}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$

multiply both sides by $(x-3)(x+2)$:

$$3x-4 = A(x+2) + B(x-3)$$

plug in $x=-2$: $\quad -10 = -5B \longrightarrow B=2$

plug in $x=3$: $\quad 5 = 5A \longrightarrow A=1$

[Could also do by equating coefficients as we did above, but plugging in is faster!]

So $\displaystyle\int = \int_0^1 x+1 + \frac{1}{x-3} + \frac{2}{x+2}\,dx = \cdots = \underline{\underline{\frac{3}{2} + \ln\frac{3}{2}}}$

What if some factor appears more than once in the denominator?

Ex $\displaystyle\int \frac{1}{x^3+2x^2+x}\,dx$

Factor: $x^3+2x^2+x = x(x^2+2x+1)$
$\qquad\qquad = x(x+1)^2$

Write $\displaystyle\frac{1}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

[Could also put $\displaystyle\frac{A}{x} + \frac{(Bx+C)}{(x+1)^2}$ — either way will work]

Mult. both sides by $x(x+1)^2$:

$$1 = A(x+1)^2 + Bx(x+1) + Cx$$
$$1 = A(x^2+2x+1) + B(x^2+x) + Cx$$

$$0x^2 + 0x + 1 = (A+B)x^2 + (2A+B+C)x + A$$

$$\longrightarrow \begin{matrix} A+B=0 \\ 2A+B+C=0 \\ A=1 \end{matrix} \longrightarrow \begin{bmatrix} A=1 \\ B=-1 \\ C=-1 \end{bmatrix}$$

Or:

$$1 = A(x+1)^2 + Bx(x+1) + Cx$$

plug in $\left.\begin{matrix} x=-1 \\ x=0 \\ x=1 \end{matrix}\right\} \to \begin{bmatrix} 1=-C \\ 1=A \\ 1=4A+2B+C \end{bmatrix} \longrightarrow \begin{bmatrix} C=-1 \\ A=1 \\ B=-1 \end{bmatrix}$