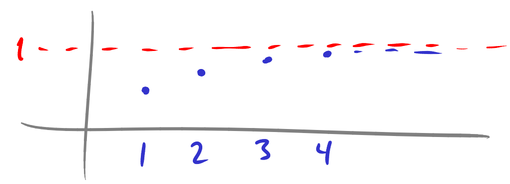
Lecture 21 | 19 Oct 2012

Last time: Sequences — ordered lists of #'s

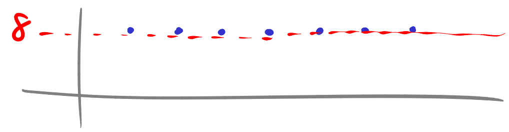
$a_1, a_2, a_3, \ldots$

Ex $a_n = \frac{n}{n+1}$: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$



(converges to 1)

$a_n = 8$: $8, 8, 8, 8, \ldots$



(converges to 8)

Ex Does $a_n = 2 + \frac{1}{n} + \frac{n!+1}{(n+1)!}$ converge? (If so, to what?)

Split it up:

$$a_n = \underset{\downarrow\, 2}{2} + \underset{\downarrow\, 0}{\frac{1}{n}} + \frac{n!}{(n+1)!} + \underset{\downarrow\, 0}{\frac{1}{(n+1)!}} \quad \left(\text{"}\frac{1}{\infty}\text{"}\right)$$

this part looks like $\frac{\infty}{\infty}$ but can't use L'Hospital here...

What is it really?

$$\frac{n!}{(n+1)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdot \cdots (1)}{(n+1)\cdot n\cdot(n-1)\cdot(n-2)\cdot \cdots (1)} = \frac{1}{n+1} \to 0$$

So $\{a_n\}$ converges to $\underline{2}$

Ex Does the sequence $a_n = \left(1 - \frac{7}{n}\right)^{-2n}$ converge? (If so, to what?)

This looks like it's $\sim 1^{-\infty}$ as $n \to \infty$.

That's an indeterminate form, (like $\frac{0}{0}$ or $\frac{\infty}{\infty}$) where we have to work harder to find the limit.

Use the following fact: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

[Why is it true? Take ln: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = L$

$$\lim_{n\to\infty} \ln\left[\left(1+\frac{x}{n}\right)^n\right] = \ln L$$

$$\lim_{n\to\infty} \underbrace{n}_{\to\infty} \ln\underbrace{\left(1+\frac{x}{n}\right)}_{\to 0} = \ln L$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{x}{n}}\cdot\left(-\frac{x}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{x}{1+\frac{x}{n}} = x$$

So $\ln L = x$, i.e. $L = e^x$]

$$a_n = \left(1-\frac{7}{n}\right)^{-2n}$$

$$= \left[\left(1-\frac{7}{n}\right)^n\right]^{-2}$$

So $\lim_{n\to\infty} a_n = \left(e^{-7}\right)^{-2} = \underline{\underline{e^{14}}}$

$$\lim_{n\to\infty}\left(1+\frac{x}{a}\right)^n = e^x$$

$$y^{ab} = (y^a)^b$$

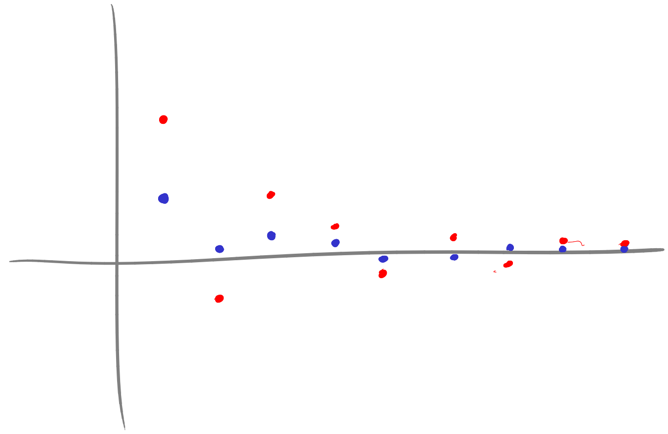
Another useful fact ("Squeeze Theorem"):

Suppose have 2 sequences a_n b_n

and b_n converges to 0

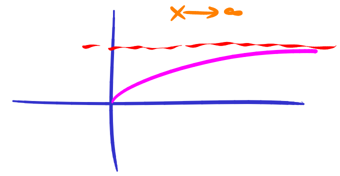
and $|a_n| < |b_n|$

Then a_n also converges to 0.



Ex $\quad a_n = \dfrac{(\tan^{-1} n)\cdot(-1)^n}{\sqrt{n}}$ $\qquad$ Does a_n converge?

$\lim\limits_{x\to\infty} \tan^{-1}(x) = \pi/2$



Intuition: as $n \to \infty$, $\quad \tan^{-1}(n) \to \pi/2$

$(-1)^n$ goes back and forth between $1, -1$

$\dfrac{1}{\sqrt{n}} \to 0$

So, we'd think $a_n \to 0$ (a_n converges to 0).

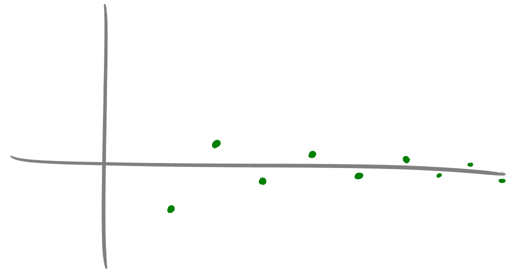
To prove it: show $|a_n| < |b_n|$ for some sequence b_n that has $b_n \to 0$

$|\tan^{-1}(n)| < \pi/2 \qquad |(-1)^n| = 1$

So consider $\quad b_n = \dfrac{\pi}{2}\cdot\dfrac{1}{\sqrt{n}} \longrightarrow 0$

We have $|a_n| < |b_n|$, so by Squeeze Theorem, $a_n \to 0$

NB: $a_n = \frac{(-1)^n}{n} = \underbrace{(-1)^n}_{\text{diverges}} \cdot \underbrace{\frac{1}{n}}_{\text{converges}}$



has $a_n \longrightarrow 0$!

So, we can't say that (divergent) $\times$ (convergent) = (divergent)

But, it is OK to say (convergent) $\times$ (convergent) = (convergent)

Ex

$$\lim_{n\to\infty} \left(\tan^{-1}(n)\right) \cdot \frac{n^2+3}{4n^2+7}$$

$$= \left(\lim_{n\to\infty} \tan^{-1}(n)\right) \cdot \left(\lim_{n\to\infty} \frac{n^2+3}{4n^2+7}\right)$$

$$= \pi/2 \cdot 1/4 = \pi/8$$