Housekeeping: Exam 2 Tue Nov 6 7-9pm

Last time: sequences $\{a_n\} = a_1, a_2, a_3, \ldots$

From now on: series — summing up sequences

Recall summation notation: e.g. $1+4+9+16+25+36$
$$= 1^2+2^2+3^2+4^2+5^2+6^2$$
$$= \sum_{i=1}^{6} i^2$$

Series (or Infinite Series) (Ch 11.2)

Take a sequence a_n. Try to take the sum of all the terms in the sequence:

Ex $a_n = n$: $1+2+3+4+5+\cdots = \sum_{n=1}^{\infty} n$

Ex $a_n = \frac{1}{2^n}$: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots = \sum_{n=1}^{\infty} \frac{1}{2^n}$

What do these sums mean?

Like we did for improper int's $\int^{\infty}$: look at the sum of k terms

$$S_k = \sum_{n=1}^{k} a_n \qquad (\text{"partial sum"})$$

Then, try to take limit as $k \to \infty$. If that exists we say the series converges, otherwise it diverges.

Ex if $a_n = n$:

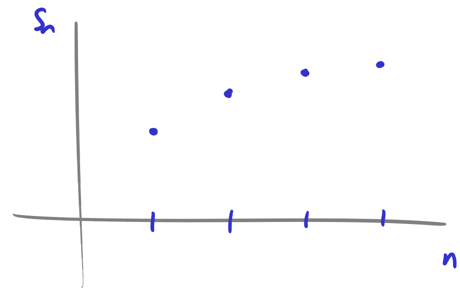
$$s_1 = 1 \qquad = 1$$
$$s_2 = 1+2 \qquad = 3$$
$$s_3 = 1+2+3 \qquad = 6$$
$$s_4 = 1+2+3+4 \qquad = 10$$
$$\vdots \qquad \qquad \vdots$$

$\{s_n\}$ doesn't converge

So, $\sum_{n=1}^{\infty} n$ doesn't converge

Ex if $a_n = \frac{1}{2^n}$:

$$s_1 = \frac{1}{2} \qquad = \frac{1}{2}$$
$$s_2 = \frac{1}{2} + \frac{1}{4} \qquad = \frac{3}{4}$$
$$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \qquad = \frac{7}{8}$$
$$s_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \qquad = \frac{15}{16}$$
$$\vdots$$



$\lim_{n\to\infty} s_n = 1$. So we say $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ (converges to 1).

NB: If we know the partial sums s_n we can recover the original seq. a_n by

$$a_n = s_n - s_{n-1}$$

Basic example: geometric series e.g. $a_n = 2, 6, 18, 54, 162, \ldots$ ($\times 3$, $\times 3$, $\times 3$, $\times 3$)

$$a_n = 2 \cdot 3^{n-1}$$

In general, $a_n = a \cdot r^{n-1}$

a: 1st term; r: ratio between successive terms

What's $\sum_{n=1}^{\infty} a \cdot r^{n-1}$?

Look at the partial sums:

$$S_k = \sum_{n=1}^{k} a \cdot r^{n-1} = a + ar + ar^2 + \cdots + ar^{k-1}$$

$$\begin{array}{rrl} & S_k = & a + ar + ar^2 + \cdots + ar^{k-1} \\ - & rS_k = & \phantom{a +{}} ar + ar^2 + \cdots + ar^{k-1} + ar^k \\ \hline & S_k - rS_k = & a \qquad\qquad\qquad\qquad\qquad - ar^k \end{array}$$

$$S_k(1-r) = a(1-r^k)$$ (if $r \neq 1$)

$$S_k = a \cdot \frac{1-r^k}{1-r}$$ (partial sums of a geometric series, if $r \neq 1$)

To get the whole sum $\sum_{n=1}^{\infty} ar^{n-1}$ we now take limit:

because r^k diverges

$$\lim_{k\to\infty} S_k = \begin{cases} \text{diverges} & \text{if } |r|>1 \\ a \cdot \dfrac{1}{1-r} & \text{if } |r|<1 \end{cases}$$

So, finally:

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} = \begin{cases} \text{diverges if } |r|>1 \\ \dfrac{a}{1-r} \text{ if } |r|<1 \end{cases}$$ (sum of a geometric series)

Ex Find the sum of the series

$$2 + \frac{1}{3} + \frac{1}{18} + \frac{1}{108} + \cdots$$

$\times\frac{1}{6}$ $\times\frac{1}{6}$ $\times\frac{1}{6}$

Geometric: $a = 2$, $r = \frac{1}{6}$

$$\text{sum} = \frac{a}{1-r} = \frac{2}{1-\frac{1}{6}} = \frac{2}{5/6} = \underline{\underline{\frac{12}{5}}}$$

Ex $4 - 3 + \frac{9}{4} - \frac{27}{16} + \cdots$

$\times -\frac{3}{4}$ $\quad \times -\frac{3}{4}$ $\quad \times -\frac{3}{4}$

Geometric: $a = 4$, $r = -\frac{3}{4}$

$$\text{Sum} = \frac{a}{1-r} = \frac{4}{1-\left(-\frac{3}{4}\right)} = \frac{4}{7/4} = \underline{\underline{\frac{16}{7}}}$$

Ex Does $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}}$ converge, and if so, to what?

First: is it geometric? Yes.

1 way to see this:

rewrite it as $$\sum 10 \cdot \frac{10^{n-1}}{(-9)^{n-1}} = \sum \underbrace{10}_{a} \cdot \underbrace{\left(-\frac{10}{9}\right)^{n-1}}_{r^{n-1}}$$

So it's geom, with $a = 10$, $r = -10/9$

$\Longrightarrow$ __does not converge__

[__Another way__ to see that the series is geometric: $a_n = \frac{10^n}{(-9)^{n-1}}$

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}/(-9)^n}{10^n/(-9)^{n-1}} = \frac{10}{-9}$$ which is just a __constant__, so it's geometric, $r = -\frac{10}{9}$]

Ex Compute the sum $\displaystyle\sum_{n=1}^{\infty} \frac{3+5^n}{7^n}$.

This is __not__ geometric, but it is the __sum__ of 2 geom series:

$$\sum_{n=1}^{\infty} \frac{3+5^n}{7^n} = \sum_{n=1}^{\infty} \frac{3}{7^n} + \sum_{n=1}^{\infty} \frac{5^n}{7^n}$$

$\left(\frac{3}{7^n} = \frac{3}{7} \cdot \frac{1}{7^{n-1}} \right)$

geom: $a = 3/7$, $r = 1/7$

geom: $a = 5/7$, $r = 5/7$

$$= \frac{3/7}{1-1/7} + \frac{5/7}{1-5/7}$$

$$= \frac{1}{2} + \frac{5}{2} = \underline{\underline{3}}$$