Lecture 23

24 Oct 2012

Last time: series $a_1 + a_2 + a_3 + \cdots$

$$= \sum_{n=1}^{\infty} a_n \qquad \left(\text{or } = \sum_{i=1}^{\infty} a_i,\ = \sum_{k=1}^{\infty} a_k, \ldots\right)$$

(Remark: $\sum_1^{\infty}$ is in many ways like $\int^{\infty}$)

[Could also have sequences beginning from $n=0$ instead of $n=1$

i.e. $a_0 + a_1 + a_2 + \cdots$

$$= \sum_{n=0}^{\infty} a_n$$]

Partial sums of $\sum_{n=1}^{\infty} a_n$ are $S_k = \sum_{n=1}^{k} a_k$

$S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$
$\vdots$

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k \quad (\text{if it exists})$$

One class of examples: geometric series $\sum_{n=1}^{\infty} a \cdot r^{n-1} = a + ar + ar^2 + \cdots$

$$\left[\text{or } \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots\right]$$

Partial sums of geom. series are $S_k = a \cdot \dfrac{1-r^k}{1-r}$

taking $k\to\infty$ limit,

$$\sum_{n=1}^{\infty} ar^{n-1} \begin{cases} \text{converges to } \dfrac{a}{1-r} & \text{if } |r|<1 \\ \text{diverges if} & |r|\geq 1 \end{cases}$$

Ex Does $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{2n+2}}$ converge?

— Is it geometric? i.e. can we write $\frac{\pi^n}{3^{2n+2}}$ as $a \cdot r^n$?

Plug in n=0: $\frac{\pi^n}{3^{2n+2}} = a \cdot r^n$

so should take $\frac{1}{9} = a$

then $$\frac{\pi^n}{3^{2n+2}} = \frac{1}{3^2} \cdot \frac{\pi^n}{3^{2n}} = \frac{1}{9}\left(\frac{\pi^n}{3^{2n}}\right) = \frac{1}{9}\left(\frac{\pi}{3^2}\right)^n = \frac{1}{9}\left(\frac{\pi}{9}\right)^n$$

So it is geometric, with $a = \frac{1}{9}$ $r = \frac{\pi}{9}$

So it converges, since $\left|\frac{\pi}{9}\right| < 1$

Ex Consider the repeating decimal

$$1.\overline{73} = 1.73737373\cdots$$

Write it as a fraction.

$$1.73737373\cdots$$

$$= 1 + \frac{73}{100} + \frac{73}{100^2} + \frac{73}{100^3} + \frac{73}{100^4} + \cdots$$

$\times\frac{1}{100}$ $\times\frac{1}{100}$ $\times\frac{1}{100}$

geom. series with $a = 73/100$

$r = 1/100$

$$= 1 + \frac{a}{1-r} = 1 + \frac{73/100}{1 - 1/100} = 1 + \frac{73/100}{99/100} = 1 + \frac{73}{99} = \frac{172}{99}$$

Ex Consider the series $\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n}$

For what values of x does it converge?

$$\sum_{n=0}^{\infty} \left(\frac{x+3}{2}\right)^n \quad \text{geometric with} \quad a = 1, \quad r = \frac{x+3}{2}$$

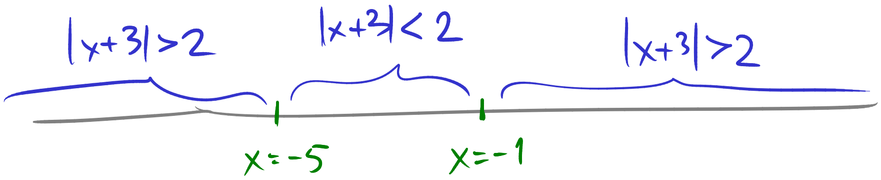
$\Longrightarrow$ it converges when $|r|<1$, i.e. $\left|\frac{x+3}{2}\right| < 1$

i.e. $|x+3| < 2$

To find boundaries of this region on the # line, solve $|x+3| = 2$

i.e. $x+3 = 2$ or $x+3 = -2$

$x = -1$ or $x = -5$



So $|x+3|<2$ means $-5 < x < -1$

Ex Consider $\sum_{n=1}^{\infty} \frac{(\cos x)^n}{3^{n+1}}$

For which values of x does this converge?

$$\sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{\cos x}{3}\right)^n \quad \text{geom:} \quad a = \frac{\cos x}{9}, \quad r = \frac{\cos x}{3}$$

(get this by plugging in $n=1$)

$$\left[= \sum_{n=1}^{\infty} \frac{\cos x}{9} \cdot \left(\frac{\cos x}{3}\right)^{n-1} \right]$$

Converges if $|r| < 1$, i.e. $\left|\frac{\cos x}{3}\right| < 1$ i.e. $|\cos x| < 3$

That's true for all x.

So $\sum$ converges for all x.

OK, enough geom. series for a while.

How about other series?

Test For Divergence / Divergence Test:

If $\lim_{n\to\infty} a_n$ doesn't exist, or if it exists but it's not 0,

then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex $1+2+3+4+\cdots = \sum_{n=1}^{\infty} n$ diverges, by Test For Divergence, b/c $\lim_{n\to\infty} n$ doesn't exist.

Ex $\sum_{n=1}^{\infty} \frac{3n+4}{4n-7}$ diverges, by TFD, b/c $\lim_{n\to\infty} \frac{3n+4}{4n-7} = \frac{3}{4} \neq 0$.

Ex $\sum_{n=1}^{\infty} \frac{1}{n^2}$: $1+\frac{1}{4}+\frac{1}{9}+\cdots$ $\lim_{n\to\infty} \frac{1}{n^2} = 0$, so TFD gives no info

$\sum_{n=1}^{\infty} \frac{1}{n}$: $\lim_{n\to\infty} \frac{1}{n} = 0$, so TFD gives no info