Lecture 25 | 29 Oct 2012

Exam 2: Tue Nov 6 7–9pm

17 problems, $\begin{cases} \sim 7\text{–}8 & \text{double } \int\text{'s} \\ \sim 9\text{–}10 & \text{seq, series} \end{cases}$

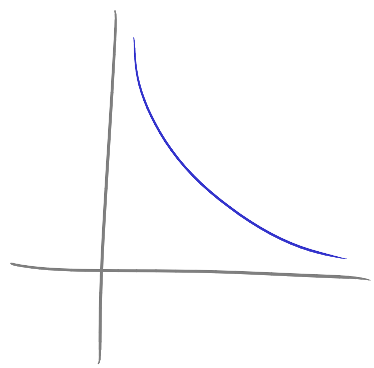
Last time: "Integral Test"

If $f(x)$ is continuous, decreasing, positive for $x > M$ (M = any #)

and $a_n = f(n)$

then $\sum_{n=M}^{\infty} a_n$ $\begin{cases} \text{converges if } \int_M^\infty f(x)\,dx \text{ converges} \\ \text{diverges if } \int_M^\infty f(x)\,dx \text{ diverges} \end{cases}$

Ex Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?



$f(x) = \frac{1}{x^2}$ is a cts, +ve, decreasing f^

So can use Integral Test:

$\int_1^\infty \frac{1}{x^2}\,dx$ converges, by p-test for integrals (look back at notes on improper $\int$) b/c $2>1$

So, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Could have done this for any $\sum_{n=1}^{\infty} \frac{1}{n^p}$. So, we have:

p-test for sums

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\le 1 \end{cases}$$

Ex Does $\sum_{k=412}^{\infty} ke^{-k}$ converge?

Try Int. Test: look at $f(x) = xe^{-x}$

Is $f(x)$ decreasing?

$$f'(x) = e^{-x} + x\cdot(-e^{-x})$$
$$= \underbrace{e^{-x}}_{+\text{ve}}\underbrace{(1-x)}_{-\text{ve as long as } x>1}$$

So $f(x)$ is decreasing for $x>1$ $\Longrightarrow$ can use Int Test.
(and $f(x)$ is positive)

So, look at $\int_{412}^{\infty} xe^{-x}\,dx = \lim_{t\to\infty} \underbrace{\int_{412}^{t} xe^{-x}\,dx}_{\text{do by } \int \text{ by parts}}$

You'll get that this limit exists, so the $\int$ converges

So, $\sum_{n=412}^{\infty} ne^{-n}$ converges by Int. Test

Other sums that you might use $\int$ Test for: eg $\sum \frac{1}{n \ln n}$

$\sum \frac{1}{n(\ln n)^2}$ ...

Comparison Tests (Ch 12.4)

Comparison Test:

Suppose we have two sequences of positive #'s a_n, b_n

1) If $\sum_{n=1}^{\infty} b_n$ is <u>convergent</u> and $a_n \le b_n$

   then $\sum_{n=1}^{\infty} a_n$ is <u>convergent</u>.

2) If $\sum_{n=1}^{\infty} b_n$ is <u>divergent</u> and $a_n \ge b_n$

   then $\sum_{n=1}^{\infty} a_n$ is <u>divergent</u>.

Ex $\sum_{n=1}^{\infty} \frac{5}{2n^3+7n^2+104}$: say $a_n = \frac{5}{2n^3+7n^2+104}$

$b_n = \frac{5}{2n^3}$

Then $a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ <u>converges</u> by p-test $(p=3>1)$

So, by Comparison Test, $\sum_{n=1}^{\infty} a_n$ <u>converges</u>.

Ex $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$ Put $a_n = \frac{\ln n}{\sqrt{n}}$ $b_n = \frac{1}{\sqrt{n}}$

$a_n \ge b_n$ (as long as $n>2$)

and $\sum_{n=1}^{\infty} b_n$ <u>diverges</u> by p-test $(\frac{1}{2}<1)$

So, $\sum_{n=1}^{\infty} a_n$ also <u>diverges</u>, by Comparison Test

Limit Comparison Test:

Suppose a_n, b_n are sequences of positive #'s and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \quad \text{with } c \neq 0$$

Then, if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

Ex Does $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converge?

Say $a_n = \frac{1}{2^n - 1}$, $b_n = \frac{1}{2^n}$. $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is geometric with $r = 1/2$, so it converges

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{2^n-1}\right)}{\left(\frac{1}{2^n}\right)} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - \underbrace{\frac{1}{2^n}}_{0}} = 1 \neq 0$$

so we can use Lim-Comp Test:

$\Longrightarrow \sum_{n=1}^{\infty} a_n$ behaves same as $\sum_{n=1}^{\infty} b_n$, i.e. converges

---

Ex Does $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{n^5 + 5}}$ converge? $\sim \frac{2n^2}{n^{5/2}} \sim 2 \cdot \frac{1}{n^{1/2}}$

First, guess: $\dfrac{2n^2+3n}{\sqrt{n^5+5}} \sim \dfrac{2n^2}{n^{5/2}} \sim \dfrac{2}{n^{1/2}}$ which diverges by p-test

so we would guess that $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{n^5+5}}$ also diverges.

To check this guess, try limit-comparison test:

let $a_n = \dfrac{2n^2+3n}{\sqrt{n^5+5}}$, $b_n = \dfrac{2}{\sqrt{n}}$. $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges by p-test.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{2n^2+3n}{\sqrt{n^5+5}}\right)}{\left(\frac{2}{\sqrt{n}}\right)} = \lim_{n\to\infty} \frac{(2n^2+3n)\sqrt{n}}{2\sqrt{n^5+5}}$$

$$= \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2n^{5/2}\sqrt{1+5n^{-5}}} = \lim_{n\to\infty} \frac{2+\overbrace{3n^{-1}}^{\to 0}}{2\sqrt{1+\underbrace{5n^{-5}}_{\to 0}}} = 1$$

So, limit-comp. test applies,

so $\displaystyle\sum_{n=1}^{\infty} a_n$ behaves same as $\displaystyle\sum_{n=1}^{\infty} b_n$, i.e. diverges.