Lecture 29 — 7 Nov 2012

Absolute Convergence (Ch 11.6)

$$\sum_{n=1}^{\infty} a_n$$

Call $\sum_{n=1}^{\infty} a_n$ "absolutely convergent" if $\sum_{n=1}^{\infty} |a_n|$ converges

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \cdots$ $\qquad a_n = \frac{(-1)^n}{n^2}$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (by p-test $p = 2 > 1$)

So $\sum a_n$ is <u>absolutely convergent</u>.

Fact: If $\sum a_n$ is absolutely convergent, then it is convergent.

If $\sum a_n$ is convergent but <u>not</u> absolutely convergent, then we call it <u>conditionally convergent</u>.

So, there are 3 possibilities.

- absolutely convergent
- Conditionally convergent
- divergent

Ex $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ <u>convergent</u> by Alt Series Test

But, $\sum_{n=1}^{\infty} \left|(-1)^n \cdot \frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ <u>divergent</u> by p-test

So, $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ is conditionally convergent

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Ex $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

Has both positive and negative terms: $+, -, -, -, +, +, +, -, \cdots$

Not alternating.

Is it absolutely convergent? Look at $\sum \left| \frac{\cos(n)}{n^2} \right|$

$$= \sum \frac{|\cos(n)|}{n^2}$$

We know $\frac{|\cos(n)|}{n^2} \leq \frac{1}{n^2}$

and $\sum \frac{1}{n^2}$ converges, so, by Comparison Test, $\sum \frac{|\cos(n)|}{n^2}$ also converges.

So, $\sum \frac{\cos(n)}{n^2}$ is absolutely convergent.

(In particular, $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ is convergent.)

Ex $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

This is an alternating series $\sum (-1)^n b_n$

with $b_n = \frac{1}{\ln n}$

b_n is decreasing, $\lim_{n\to\infty} b_n = 0$, so by Alt. Series Test,

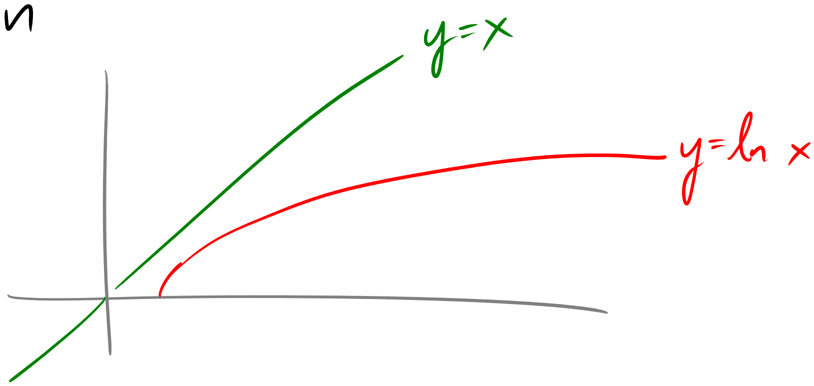
$\sum \frac{(-1)^n}{\ln n}$ converges.

Does it converge absolutely?

$\sum \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ — does this Σ converge?

$$\frac{1}{\ln n} > \frac{1}{n}$$

[because $\ln n < n$



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and $\sum \frac{1}{n}$ diverges, so, by Comp Test, $\sum \frac{1}{\ln n}$ also diverges.

So, $\sum (-1)^n \frac{1}{\ln n}$ converges conditionally.

Ratio Test

1) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

2) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ (or $=\infty$) then $\sum_{n=1}^{\infty} a_n$ is divergent.

(If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ then the test is inconclusive.)

Ex $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$.

Ratio test:

$$a_n = (-1)^n \frac{n^3}{3^n}$$

$$a_{n+1} = (-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\left[\frac{(n+1)^3}{3^{n+1}}\right]}{\left[\frac{n^3}{3^n}\right]} = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3^{n+1}}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{3}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^3 \cdot \frac{1}{3} = \frac{1}{3}$$

So $L = \frac{1}{3} < 1$, so $\sum_1 (-1)^n \frac{n^3}{3^n}$ <u>converges</u> by Ratio Test