Lecture 33 | 16 Nov 2012

Last time: power series $\sum c_n (x-a)^n$

Functions As Power Series (Ch 11.9)

Using the formula for $\sum$ of a geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$(\text{for } |x|<1)$

[geom. series
first term = 1
common ratio = x]

i.e. $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad (\text{for } |x|<1)$$

Ex $$\frac{1}{0.7} = \frac{1}{1-0.3} = 1 + 0.3 + (0.3)^2 + (0.3)^3 + (0.3)^4 + \cdots$$
$$= 1 + 0.3 + 0.09 + 0.027 + 0.0081 + \cdots$$
$$\approx 1.43$$

Ex Find a representation of the function $\frac{1}{1+x^2}$ as a power series centered at 0, and find its interval of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \frac{1}{1-y} \qquad y = -x^2$$

and we know $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$

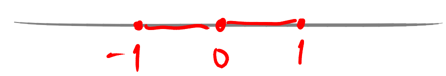
So $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

i.e. $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$

To find interval of convergence: could use Ratio Test.
But here there's a shortcut: our series is <u>geometric</u> with $r = -x^2$.
So it converges <u>if + only if</u>

$$|r| < 1$$
$$|-x^2| < 1$$
$$|x|^2 < 1$$
$$|x| < 1$$

So interval of convergence is $(-1, 1)$. Radius of conv. $R = 1$.



<u>Ex</u> Write $\dfrac{1}{x+7}$ as a power series centered at 0.

$$\frac{1}{x+7} = \frac{1}{7}\left(\frac{1}{\frac{x}{7}+1}\right) = \frac{1}{7}\left(\frac{1}{1-\left(-\frac{x}{7}\right)}\right)$$

$$= \frac{1}{7}\left(\sum_{n=0}^{\infty}\left(-\frac{x}{7}\right)^n\right)$$

$$= \frac{1}{7}\sum_{n=0}^{\infty}\frac{(-1)^n}{7^n}x^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{7^{n+1}}x^n$$

Remark: We could also have written

$$\frac{1}{x+7} = \frac{1}{1-(-x-6)} = \sum_{n=0}^{\infty} (-x-6)^n = \sum_{n=0}^{\infty} (-1)^n (x+6)^n$$

This is also perfectly fine series, centered around $x=-6$ instead of $x=0$

Ex Write the function $\frac{x^4}{x+7}$ as a power series centered at 0

from the last example

$$\frac{x^4}{x+7} = x^4 \cdot \frac{1}{x+7} = x^4 \cdot \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{7^{n+1}} x^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{7^{n+1}} x^{n+4}$$

Could also rewrite this: let $m = n+4$ so $n = m-4$

$$= \sum_{m=4}^{\infty} \frac{(-1)^{m-4}}{7^{m-3}} x^m$$

$$= \sum_{m=4}^{\infty} \frac{(-1)^m}{7^{m-3}} x^m$$

How about $\frac{1}{(1-x)^2}$?

Trick: We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

Take $\frac{d}{dx}$ of both sides:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$$

i.e. $$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

Fact: If we have a power series for $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Then also:

$$\frac{d}{dx} f(x) = \frac{d}{dx}\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} c_n \cdot n (x-a)^{n-1}$$

$$\int f(x)\,dx = \int \sum_{n=0}^{\infty} c_n (x-a)^n\,dx = \left(\sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}\right) + C$$

Both these new series have the <u>same</u> radius of convergence as the original one.

<u>Ex</u> Express $\ln(1-x)$ as a power series centered at 0.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\int \frac{1}{1-x}\,dx = \int \sum_{n=0}^{\infty} x^n\,dx$$

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$

$$\ln(1-x) = \sum_{n=0}^{\infty}\left(-\frac{x^{n+1}}{n+1}\right) + C$$

To determine C: plug in $x=0$, then this becomes

$$\ln(1) = 0 + C$$

$$0 = 0 + C$$

so $\quad C = 0$

$$\ln(1-x) = \sum_{n=0}^{\infty}\left(-\frac{x^{n+1}}{n+1}\right)$$

or, alternatively, setting $m = n+1$, $\ln(1-x) = \sum_{m=1}^{\infty}\left(-\frac{x^m}{m}\right)$

i.e. $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

Another method (trickier) for $\frac{1}{(1-x)^2}$:

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)^2 = (1+x+x^2+x^3+\cdots)^2$$

$$= (1+x+x^2+x^3+\cdots)(1+x+x^2+x^3+\cdots)$$

$$= 1+2x+3x^2+4x^3+\cdots$$
