# Lecture 34

19 Nov 2012

Housekeeping: no lecture Wednesday

Last time: functions as power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x|<1 \qquad (*)$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots \quad \text{for } |x|<1 \qquad (**)$$

by subst. $x \to -x^2$ in $(*)$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \cdots \quad \text{for } |x|<1$$

by taking $\frac{d}{dx}$ of both sides of $(*)$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \quad \text{for } |x|<1$$

by taking $\int dx$ of both sides of $(*)$

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Integrating both sides of $(**)$:

$$\int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

and plug in $x=0$: $\quad 0 = 0 + C \quad$ so $\quad C = 0$

i.e. $$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

Cute application: plug in $x=1$, get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

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<u>Ex</u> Find the power series representing $f(x) = \ln\left(\frac{1+x}{1-x}\right)$ centered at $x=0$.

$$f(x) = \ln(1+x) - \ln(1-x)$$

and already know $\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$

so $\ln(1+x) = \ln(1-(-x)) = -\sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1} = -\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

$$\ln(1+x) - \ln(1-x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right)$$

$$= 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \cdots$$

$$= \underline{\underline{2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}}}$$

<u>or</u>: $\ln(1+x) - \ln(1-x) = \left(-\sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1}\right) - \left(-\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}\right)$

$$= -\sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1} - \frac{x^{n+1}}{n+1}$$

$$= -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \cdot \left((-1)^{n+1} - 1\right)$$

$$= -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \cdot \begin{cases} 0 & \text{if } n \text{ odd} \\ -2 & \text{if } n \text{ even} \end{cases}$$

write even terms, let $n=2m$:
$$= -\sum_{m=0}^{\infty} (-2)\cdot \frac{x^{2m+1}}{2m+1}$$
$$= 2\sum_{m=0}^{\infty} \frac{x^{2m+1}}{2m+1}$$

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**Remark:** We said $\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

Can also put $m=n+1$, then $\ln(1-x) = \sum_{m=1}^{\infty} \frac{x^m}{m}$

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How do we get power series representing a more general function $f(x)$?

## Taylor (and Maclaurin) Series (Ch 11.10)

If we have any function $f(x)$ which is "nice enough" (can be differentiated as many times as we want) and any number $a$, we can write down the Taylor series of $f$ centered at $a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \text{power series centered at } a$$

$$= f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2 + \tfrac{1}{6}f'''(a)(x-a)^3 + \cdots$$

If this series has radius of conv. $R>0$ then its sum is $f(x)$ for $x \in (a-R, a+R)$

i.e. $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{for } x\in(a-R, a+R)$$

Ex Find the Taylor series for $f(x) = e^x$ around $x = 0$ and its radius of convergence.

Taylor series: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

and $f^{(n)}(x) = e^x$ for all $n$, so $f^{(n)}(0) = 1$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

Radius of conv: use Ratio Test, find $R = \infty$