Lecture 35

26 Nov 2012

Housekeeping: Exam 3 Dec 4 (next Tue) 7-9pm

Exam review Mon Dec 3

Final exam review Wed Dec 5, Fri Dec 7

Some older material:

Ex What is the function represented by the power series

$$\sum_{n=0}^{\infty} (-1)^n \cdot 2n\, x^{2n-1} \;?$$

Notice, $2n\, x^{2n-1} = \frac{d}{dx}(x^{2n})$

So
$$\sum_{n=0}^{\infty} (-1)^n\, 2n\, x^{2n-1} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n x^{2n}\right)$$
$$= \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-x^2)^n\right)$$
$$= \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \underline{\frac{-2x}{(1+x^2)^2}}$$

Last time: first glimpse of Taylor series

Taylor series of a function $f(x)$ centered at a is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

If this series has r.o.c. R, then for any x with $|x-a|<R$ the series converges $\underline{\underline{\text{to } f(x)}}$ i.e.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \cdots$$

If $a=0$ we also call this series Maclaurin series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Ex Find the Maclaurin series for $f(x) = e^x$ and its rad. of conv.

Macl: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$$\begin{aligned} f(x) &= e^x & f(0) &= e^0 = 1 \\ f'(x) &= e^x & f'(0) &= e^0 = 1 \\ f''(x) &= e^x & f''(0) &= 1 \\ &\vdots \\ f^{(n)}(x) &= e^x & f^{(n)}(0) &= 1 \end{aligned}$$

So, the Maclaurin series for $f(x) = e^x$ is $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

Radius of conv: use Ratio Test $\qquad a_n = \frac{x^n}{n!}$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)!} \frac{n!}{|x|^n} = \frac{|x|}{n+1} \longrightarrow 0 \text{ as } n \to \infty$$

since $0 < 1$, this means the Σ converges for all x i.e. $R = \infty$

So $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for all x

(In particular, plug in $x=1$: $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$)

Ex Find the Maclaurin series for $f(x) = \sin x$ and its radius of convergence.

$f(x) = \sin x$	$f(0) = \sin(0) = 0$
$f'(x) = \cos x$	$f'(0) = \cos(0) = 1$
$f''(x) = -\sin x$	$f''(0) = -\sin(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -\cos(0) = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = \sin(0) = 0$
$\vdots$	$\vdots$
(repeats with period 4)	(repeats with period 4)

Macl series

$$\cancel{f(0)}^{0} + \frac{f'(0)}{1!}x + \cancel{\frac{f''(0)}{2!}}^{0}x^2 + \frac{f'''(0)}{3!}x^3 + \cancel{\frac{f^{(4)}(0)}{4!}}^{0}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cancel{\frac{f^{(6)}(0)}{6!}}^{0}x^6 + \frac{f^{(7)}(0)}{7!}x^7 + \cdots$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

What is rad. of conv.? Like in prev. example, use ratio test: get $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$ for all x, so $\underline{R = \infty}$.

So $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x.

Ex Find the first 3 terms of the Taylor series for $f(x) = \frac{1}{\sqrt{x}}$ centered at $a = 9$.

$$f(x) = x^{-1/2} \qquad f(9) = \frac{1}{3}$$

$$f'(x) = -\frac{1}{2}x^{-3/2} \qquad f'(9) = -\frac{1}{54}$$

$$f''(x) = \frac{3}{4}x^{-5/2} \qquad f''(9) = \frac{1}{324}$$

First 3 terms of Taylor series:

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \qquad a=9$$

$$= \underline{\frac{1}{3} - \frac{1}{54}(x-9) + \frac{1}{648}(x-9)^2}$$

This is also called the "Taylor polynomial for $f(x)$, centered at $a=9$, of degree 2."

Ex Find the Maclaurin series for $f(x) = x^3\sin(x)$.

We could use the formula for Macl. series, but that's a pain.

Trick: we already know the Macl. series for $\sin(x)$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x$$

$$\text{So } x^3\sin(x) = x^3\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = \underline{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+4}}{(2n+1)!}} \quad \text{for all } x.$$

Ex Find the Maclaurin series for $f(x) = \cos(x)$.

Trick: $\cos(x) = \frac{d}{dx}\sin(x)$

$$= \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right) \quad \text{for all } x$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x$$

So, $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \underline{1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots}$