Lecture 37

30 Nov 2012

Housekeeping:

Exam 3 next Tue 7–9pm   19 problems

tip: memorize series for e^x, $\sin x$, $\cos x$, $\frac{1}{1-x}$, $\ln(1-x)$, $\tan^{-1} x$

HW 15 due next Tue 3am

Last time: Taylor series / Taylor polynomials

Taylor series for $f(x)$:

$$f(x) = \overbrace{\underbrace{f(a) + f'(a)(x-a)}_{\text{Taylor poly. of degree 1 for } f(x)} + \frac{f''(a)}{2}(x-a)^2}^{\text{Taylor poly. of degree 2 for } f(x)} + \frac{f'''(a)}{6}(x-a)^3 + \cdots$$

Often use Taylor poly. to estimate things we can't calculate exactly, such as value of $f(x)$ at x near a, or the integral of $f(x)$ over some region near a.

Ex A car crosses the finish line with speed 76 m/s (= 170 mph) decelerating at rate 6 m/s².

Estimate how far the car goes in the 10 s after it crosses the line, using a Taylor polynomial.

We consider the function $s(t)$ = position of car at time t $s(0) = 0$

We want to estimate $s(10\text{ s})$

We know $s(0) = 0$, $s'(0) = 76$ m/s, $s''(0) = -6$ m/s²

So, we can write <u>Taylor polynomial</u>:

$$T_2(t) = s(0) + s'(0)\cdot t + \frac{s''(0)}{2}\cdot t^2$$

$$= 0 + (76\,m/s)\cdot t + \frac{(-6\,m/s^2)}{2}\,t^2$$

$$= (76\,m/s)\cdot t - (3\,m/s^2)\cdot t^2$$

$$T_2(10\,s) = (76\,m/s)\cdot(10\,s) - (3\,m/s^2)\cdot(10\,s)^2$$

$$= 760\,m - 300\,m$$

$$= \underline{460\,m}$$

<u>Ex</u> Use Taylor series to determine

$$\lim_{x\to 0} \frac{\sin(x) - x\cos(x)}{x^3}.$$

We could do this by L'Hospital, but Taylor is a better way of thinking about it...

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \qquad \text{for all } x$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots \qquad \text{for all } x$$

So
$$\lim_{x\to 0} \frac{\sin(x) - x\cos(x)}{x^3} = \lim_{x\to 0} \frac{\left(x - \frac{x^3}{6} + \cdots\right) - x\left(1 - \frac{x^2}{2} + \cdots\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{6} + \cdots\right) - \left(x - \frac{x^3}{2} + \cdots\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\left(-\frac{x^3}{6} + \frac{x^3}{2} + \cdots\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\left(\frac{1}{3}x^3 + \cdots\right)}{x^3} = \lim_{x\to 0} \frac{1}{3} + \cdots$$

positive powers of x

$$= \frac{1}{3}$$

Ex Suppose $f(x)$ has

$$f(1) = 4$$
$$f'(1) = -8$$
$$f''(1) = 24$$
$$f'''(1) = 120$$

Estimate $f(1.1)$.

Taylor poly: $T_3(x) = f(1) + f'(1)\cdot(x-1) + f''(1)\cdot\frac{1}{2}(x-1)^2 + f'''(1)\cdot\frac{1}{6}(x-1)^3 + \cdots$

$$= 4 + (-8)(x-1) + 12(x-1)^2 + 20(x-1)^3$$

$$f(1.1) \approx T_3(1.1) = 4 + (-8)(.1) + 12(.01) + 20(.001)$$
$$= 4 - .8 + .12 + .02$$
$$= 3.34$$

Estimate $f'(1.1)$.

For this, need Taylor poly for f' rather than for f.

Get it by applying $\frac{d}{dx}$ to Taylor poly for f:

$$\frac{d}{dx}\left(4 + (-8)(x-1) + 12(x-1)^2 + 20(x-1)^3\right)$$
$$= -8 + 24(x-1) + 60(x-1)^2$$

Plug in $x = 1.1$:

$$f'(1.1) \approx -8 + 24(.1) + 60(.01)$$
$$= -8 + 2.4 + .6$$
$$= \underline{\underline{-5}}$$

<u>Something cool:</u>

Recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \cdots$

Recall <u>imaginary unit</u> i $\qquad i^2 = -1$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots$$
$$= 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + \cdots$$
$$= \cos x + i \sin x$$

i.e. $\boxed{e^{ix} = \cos x + i \sin x}$ "Euler's formula"

e.g. plug in $x = \pi$: $e^{i\pi} = \cos(\pi) + i\sin(\pi)$

i.e. $\boxed{e^{i\pi} = -1}$

Euler's formula leads to familiar trig identities:

Ex

$$e^{ix} = \cos x + i \sin x$$

$$e^{2ix} = (\cos x + i \sin x)^2 = \cos^2 x + 2i \sin x \cos x - \sin^2 x$$

$$\shortparallel$$

$$\cos 2x + i \sin 2x$$

So,

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\sin 2x = 2 \sin x \cos x$$