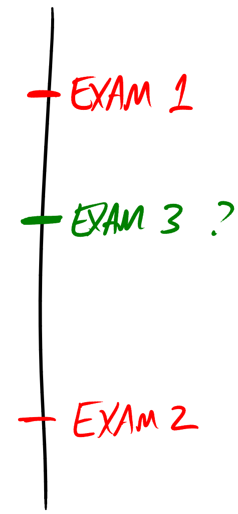
Exam 3 tomorrow 7–9pm

19 problems:
- 9 convergence (incl. interval of convergence)
- 10 power series $\begin{cases} 5 \text{ have word "Taylor"} \\ 5 \text{ don't have word "Taylor"} \end{cases}$

Difficulty:



Things to review:

- Basic tests for convergence – esp. ratio test, root test

Ratio test: calculate $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

Root test: $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

If $L<1$, converges

$L>1$, diverges

Ex $\sum 4\left(2+\frac{1}{n}\right)^{3n}$

root test: $\sqrt[n]{|a_n|} = \underbrace{\sqrt[n]{4}}_{\to 1} \cdot \underbrace{\left(2+\frac{1}{n}\right)^3}_{\to 8} \Rightarrow$ diverges

Ex $\sum \frac{(n-1)}{(n!)^2}$ ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{n}{((n+1)!)^2} \cdot \frac{(n!)^2}{(n-1)}$$

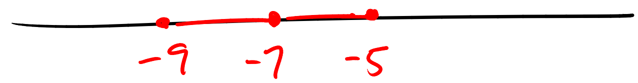
$$= \frac{n}{n-1} \cdot \left(\frac{n!}{(n+1)!}\right)^2$$

$$= \underbrace{\frac{n}{n-1}}_{\to 1} \cdot \underbrace{\left(\frac{1}{n+1}\right)^2}_{\to 0} \Rightarrow \text{converges}$$

Intervals of convergence

Ex $\sum \frac{(x+7)^n}{2^n+1}$

Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x+7|^{n+1}}{2^{n+1}+1}\,\frac{2^n+1}{|x+7|^n}$



$$= |x+7|\cdot\frac{2^n+1}{2^{n+1}+1} \longrightarrow \frac{1}{2}|x+7|$$

$$\frac{1}{2}|x+7|<1 \Rightarrow |x+7|<2$$

So radius of convergence $= 2$

To find interval of conv, plug in $x=-9$ and $x=-5$.

$x=-5$: $\sum \frac{2^n}{2^n+1}$ diverges by TFD

$x=-9$: $\sum \frac{(-2)^n}{2^n+1}$ div by TFD

So int of conv. is $(-5,-9)$

Alt. Series Test: $\sum (-1)^n b_n$ converges if $\begin{cases} b_n \text{ is decreasing} \\ \lim_{n\to\infty} b_n = 0 \end{cases}$

Doesn't apply to $\sum (-1)^n \frac{2^n}{2^n+1}$

Does apply to $\sum (-1)^n \frac{1}{n^{2/3}+1}$ — so $\sum (-1)^n \frac{1}{n^{2/3}+1}$ converges (conditionally)

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$$

Lim-Comp Test: If $\lim_{n\to\infty}\frac{a_n}{b_n} = L \qquad L \neq 0, \infty$

then $\sum a_n$, $\sum b_n$ either <u>both</u> <u>conv</u> or <u>both</u> <u>div</u>

Substitution of series: use series for f^n you already know to get harder ones,

e.g.

$$x^3\tan^{-1}x = x^3\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+4}}{2n+1}$$

$$\frac{1}{4-3x} = \frac{1}{4}\left(\frac{1}{1-\frac{3}{4}x}\right) = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{3}{4}x\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{3^n}{4^{n+1}}x^n$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x}$$

so get series for $\frac{1}{(1-x)^2}$ by taking $\frac{d}{dx}$ of series for $\frac{1}{1-x}$:

$$\left[\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1-x)^{-1} = (1-x)^{-2}\right]$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}nx^{n-1} \leftarrow$$

NB: the first term here, with $n=0$, is $0\cdot x^{0-1}$

So this is also

$$= \sum_{n=1}^{\infty}nx^{n-1}$$

$$\frac{1}{(1-3x)^2} = \sum_{n=0}^{\infty} n(3x)^{n-1} = \sum_{n=0}^{\infty} n 3^{n-1} x^{n-1}$$

$$\frac{1}{(6-4x)^2} = \frac{1}{6^2}\frac{1}{(1-\frac{2}{3}x)^2} = \frac{1}{6^2}\sum_{n=0}^{\infty} n\left(\frac{2}{3}x\right)^{n-1}$$

Shifting index:

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} x^{n+2}$$

can also be rewritten: $m = n+2 \qquad n = m-2$

$$\sum_{m=2}^{\infty} \frac{1}{(m-2)^2+1} x^m$$

$$\sum \frac{1}{n^2+\ln(n)+17} \quad \text{conv. by comparison to } \frac{1}{n^2}$$

$$\sum \frac{1}{n^{\ln(n)}} \quad \text{conv. by comp. to } \frac{1}{n^2}$$

(absolutely)

Finding r.o.c:

$$\left|\frac{2x-1}{2}\right| < 1$$

$$\left|x-\frac{1}{2}\right| < 1 \qquad R=1$$
