Lecture 5

29 Jan 2010

Reminder: my office hours MF 1:30–2:30 RLM 9.134

Last time: definition of $\int_a^b f(x)\,dx$

Now we learn a much easier way to calculate integrals. (Sec 5.3)

Fundamental Theorem of Calculus I:

If $F(x) = \int_a^x f(t)\,dt$

then $F'(x) = f(x)$. [ie $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.]

Examples. • What is the derivative of $F(x) = \int_{-4}^{x} \sin t\,dt$?

By FTCI, $F'(x) = \sin x$.

• What is the derivative of $F(x) = \int_4^{x^2} \cos t\,dt$?

[Careful – not just $\cos(x^2)$!]

Apply chain rule: $\frac{d}{dx} \int_4^{x^2} \cos t\,dt$ $\quad u = x^2$

$$= \frac{d}{dx} \int_4^{u} \cos t\,dt$$

$$= \frac{du}{dx} \cdot \frac{d}{du} \int_4^{u} \cos t\,dt$$

$$= 2x \cdot \cos(u)$$

$$= 2x \cdot \cos(x^2)$$

- Suppose $\int_{-1}^{x} f(t)\,dt = \frac{1}{x^2+1}$. What is $f(2)$?

Use FTC I: apply $\frac{d}{dx}$ to both sides.

$$\frac{d}{dx}\int_{-1}^{x} f(t)\,dt = \frac{d}{dx}\frac{1}{x^2+1}.$$

$$f(x) = -\frac{2x}{(x^2+1)^2}$$

$$f(2) = -\frac{4}{25}$$

Ex $\frac{d}{dx}\int_{x}^{5} f(x)\,dx = \frac{d}{dx}\left(-\int_{5}^{x} f(x)\,dx\right) = -f(x)$

Fundamental Theorem of Calculus II:

$\int_{a}^{b} f(x)\,dx = F(b) - F(a)$ where F is any antiderivative of f.

notation: $F(b)-F(a)$ is also written as $F\Big|_a^b$ (or $F\Big]_a^b$)

[Exercise: try to derive this from FTC I!]

Examples:

- Calculate $\int_0^1 x^2\,dx$.

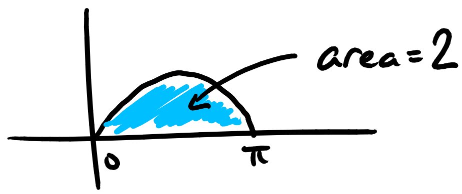
Use FTC II: $F(x) = \frac{1}{3}x^3$ is an antideriv. of x^2, so

$$\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}(1^3) - \frac{1}{3}(0^3) = \frac{1}{3}$$

- Calculate $\int_0^{\pi} \sin x \, dx$.

$F(x) = -\cos x + C$ is an antideriv. of $\sin x$, so

$$\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = (-\cos \pi + C) - (-\cos 0 + C)$$
$$= -(-1) - (-1) + C - C$$
$$= 1 + 1 = \underline{\underline{2}}$$



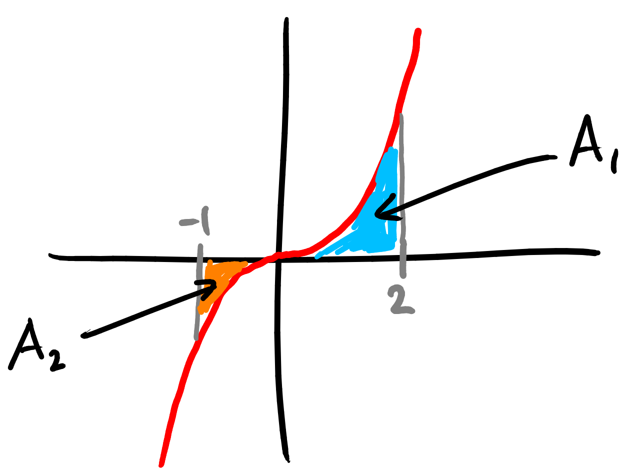
- Calculate $\int_{\pi/4}^{\pi/3} \sec\theta \tan\theta \, d\theta$.

$\sec\theta$ is an antiderivative of $\sec\theta\tan\theta$, so

$$\int_{\pi/4}^{\pi/3} \sec\theta \tan\theta \, d\theta = \sec\theta \Big|_{\pi/4}^{\pi/3} = \sec\frac{\pi}{3} - \sec\frac{\pi}{4}$$
$$= \underline{\underline{2 - \sqrt{2}}}$$

- Calculate $\int_{-1}^{2} x^3 \, dx$ and interpret it as a <u>difference</u> of <u>areas</u>.

$$\int_{-1}^{2} x^3 \, dx = \frac{x^4}{4} \Big|_{-1}^{2} = \frac{2^4}{4} - \frac{(-1)^4}{4} = \frac{15}{4}.$$



$$A_1 - A_2 = \frac{15}{4}$$

• Calculate $\int_{\pi/6}^{\pi/3} \left(-\frac{3}{\sin^2\theta} + \theta\right) d\theta$.

$$= \int_{\pi/6}^{\pi/3} \left(-3\csc^2\theta + \theta\right) d\theta$$

$$= 3\cot\theta + \frac{\theta^2}{2} \Big|_{\pi/6}^{\pi/3}$$

$$= \cdots = -2\sqrt{3} + \frac{\pi^2}{24}$$

• Calculate $\int_{1}^{-2} 3 + u^4 \, du$

$$= 3u + \frac{1}{5}u^5 \Big|_{1}^{-2}$$

$$= \left[3(-2) + \frac{1}{5}(-2)^5\right] - \left[3(1) + \frac{1}{5}(1)^5\right]$$

$$= -6 - \frac{32}{5} - 3 - \frac{1}{5} = -\frac{78}{5}$$

An example that belongs in the previous lecture:

• If $\int_1^3 f(x)\, dx = 4$

and $\int_3^7 f(x)\, dx = 16$

what is $\int_1^7 3f(x)\, dx$?

$$= 3\int_1^7 f(x)\, dx$$

$$= 3\left(\int_1^3 f(x)\, dx + \int_3^7 f(x)\, dx\right)$$

$$= 3(4+16) = \underline{\underline{60}}$$