Lecture 7 | 3 Feb 2010

Last few days: definite and indefinite integrals

Today:

Method of substitution ("u-substitution") (Ch 5.5)

Ex $\int \sqrt{2x-3}\, dx = ?$

Try to get this related to s.t. simple that we already understand: introduce

$$u = 2x-3$$

Replace x by u everywhere.

$$\int \sqrt{2x-3}\, dx = \int \sqrt{u}\, dx$$

To relate dx to du: $\frac{du}{dx} = 2$, so $du = 2\,dx$

so $\frac{1}{2} du = dx$

$$\text{so } \int \sqrt{u}\, dx = \int \sqrt{u}\, \tfrac{1}{2}\, du$$
$$= \tfrac{1}{2} \int \sqrt{u}\, du$$
$$= \tfrac{1}{2} \cdot \tfrac{2}{3} u^{3/2} + C$$
$$= \tfrac{1}{3} u^{3/2} + C = \underline{\tfrac{1}{3} (2x-3)^{3/2}} + C$$

Ex $\int 7x e^{x^2}\, dx = ?$

Set $u = x^2$.

Then $e^{x^2} = e^u$

and $\frac{du}{dx} = 2x \implies du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$$\int 7x e^{x^2}\,dx = 7\int e^{u}(x\,dx) = 7\int e^{u}\cdot\tfrac{1}{2}\,du$$
$$= \tfrac{7}{2}\int e^{u}\,du$$
$$= \tfrac{7}{2}e^{u} + C$$
$$= \underline{\underline{\tfrac{7}{2}e^{x^2} + C}}$$

Substitution Rule: If $u = g(x)$
then $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du.$

<u>Ex</u> $\displaystyle\int \frac{x^2+16x+8}{\sqrt{\frac{x}{2}+1}}\,dx = ?$

Set $u = \frac{x}{2}+1.$

$\Rightarrow$ $x = 2u-2$
$dx = 2\,du$

$$= \int \frac{(2u-2)^2+16(2u-2)+8}{\sqrt{u}}\cdot 2\,du$$

$$= 2\int \frac{4u^2-8u+4+32u-32+8}{\sqrt{u}}\,du$$

$$= 2\int 4u^{3/2}+24u^{1/2}-20u^{-1/2}\,du$$

$$= 2\left(4\cdot\tfrac{2}{5}u^{5/2}+24\cdot\tfrac{2}{3}u^{3/2}-20\cdot 2u^{1/2}\right)+C$$

$$= \tfrac{16}{5}u^{5/2}+32u^{3/2}-80u^{1/2}+C$$

and substitute back $u = \frac{x}{2}+1$ ...

[final result $= \underline{\underline{\tfrac{4}{5}\sqrt{\tfrac{x}{2}+1}\left(x^2+24x-56\right)}}$]

Substitution for <u>definite</u> integrals: Remember to transform the <u>endpoints</u>!

<u>Ex</u> $\int_0^{\pi/2} \sin(2x)\,dx$

$u = 2x$

$\frac{du}{dx} = 2 \Rightarrow du = 2\,dx$

$dx = \frac{1}{2}du$

<u>endpoints</u>: $x=0$ becomes $u=0$

$x = \frac{\pi}{2}$ becomes $u = \pi$

$= \int_0^{\pi} \sin(u) \cdot \frac{1}{2}\,du$

$= \frac{1}{2}\left(-\cos(u)\right)\Big]_0^{\pi}$

$= \frac{1}{2}\Big[[-(-1)] - [-1]\Big]$

$= \frac{1}{2}(1+1) = \underline{\underline{1}}$

<u>Ex</u> $\int_{\pi/3}^{\pi/2} \cos 3x \; e^{\sin 3x}\,dx$

$u = \sin 3x$

$\frac{du}{dx} = 3\cos 3x$

$du = 3\cos 3x\,dx$

$dx = \frac{du}{3\cos 3x}$

<u>Limits</u>: $x = \frac{\pi}{3}$ is $u = \sin \pi = 0$

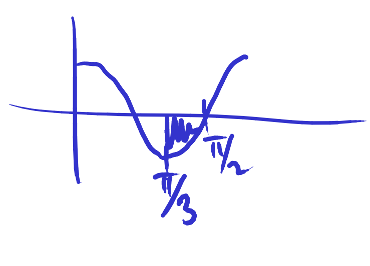
$x = \frac{\pi}{2}$ is $u = \sin\frac{3\pi}{2} = -1$

$= \int_0^{-1} \cos 3x \; e^u \frac{du}{3\cos 3x}$

$= \frac{1}{3}\int_0^{-1} e^u\,du$

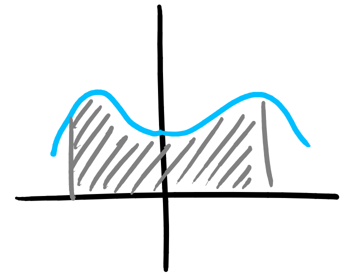
$= \frac{1}{3}\left[e^u\Big|_0^{-1}\right]$

$= \underline{\underline{\frac{1}{3}\left[e^{-1} - 1\right]}}$



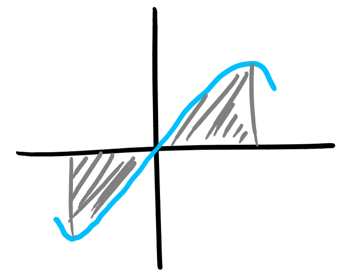
Integrals of symmetric functions:

a) If f is even $\qquad f(x) = f(-x)$



$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

b) If f is odd $\qquad f(-x) = -f(x)$



$$\int_{-a}^{a} f(x)\,dx = 0$$

Ex

$$\int_{-0.154}^{0.154} \frac{\overbrace{(\tan x)}^{\text{odd}}\overbrace{\left(x^6 + 29x^4 + \frac{105}{3}x^2 + 981.2\right)}^{\text{even}}}{\underbrace{x^{12} + 77x^6 + \cos(384x)}_{\text{even}}}\,dx = \underline{\underline{0}}$$

$$\left[\frac{\text{odd}\cdot\text{even}}{\text{even}} = \text{odd}\right]$$