Lecture 8

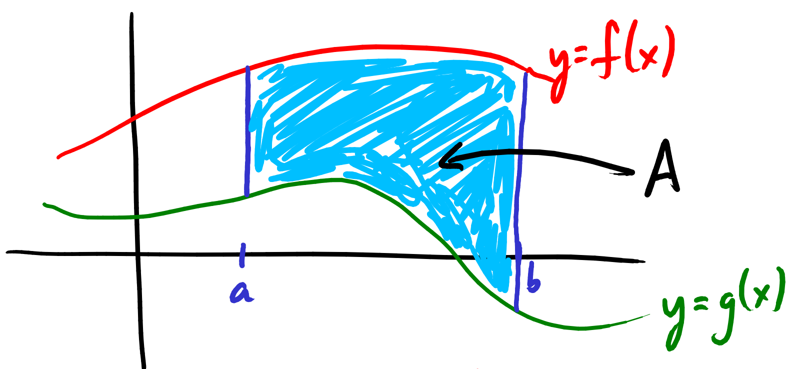
5 Feb 2010

Housekeeping:

Notes available at http://www.ma.utexas.edu/users/neitzke
(+ first day handout)

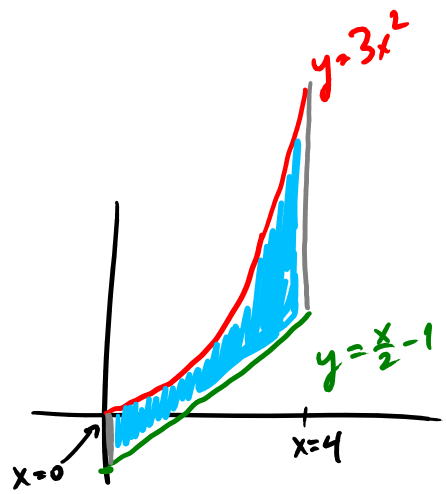
Areas between curves (Ch 6.1)

Two curves $y=f(x)$ and $y=g(x)$. Suppose $f(x)>g(x)$ for x in $[a,b]$.



The area A is given by $\int_a^b (f(x)-g(x))\,dx$.

Example. Find the area between the curves $y=3x^2$ and $y=\frac{x}{2}-1$ from $x=0$ to $x=4$.



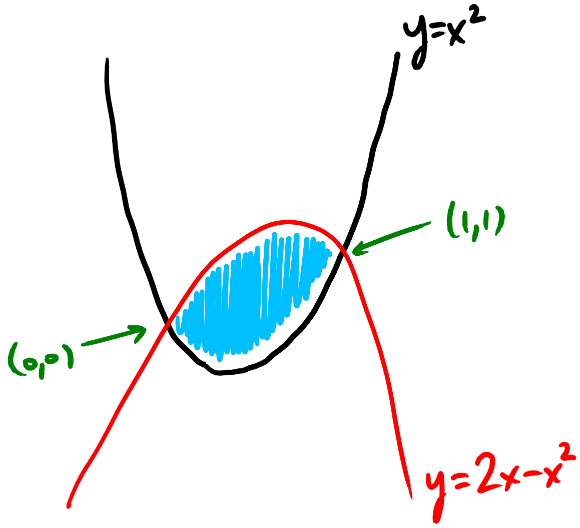
$$A=\int_0^4 (3x^2)-\left(\frac{x}{2}-1\right)\,dx$$

$$=\int_0^4 3x^2-\frac{x}{2}+1\,dx$$

$$= x^3 - \frac{1}{4}x^2 + x \Big|_0^4$$

$$= 64 - 0 = \underline{\underline{64}}$$

__Example.__ Find the area of the region between $y = x^2$ and $y = 2x - x^2$.



Start by finding the points of intersection:

$$x^2 = 2x - x^2$$

$$2x^2 - 2x = 0$$

$$2(x)(x-1) = 0$$

So the intersections occur at $x = 0$ $[y=0]$ and $x = 1$. $[y=1]$

$$A = \int_0^1 \left[(2x - x^2) - x^2\right] dx$$

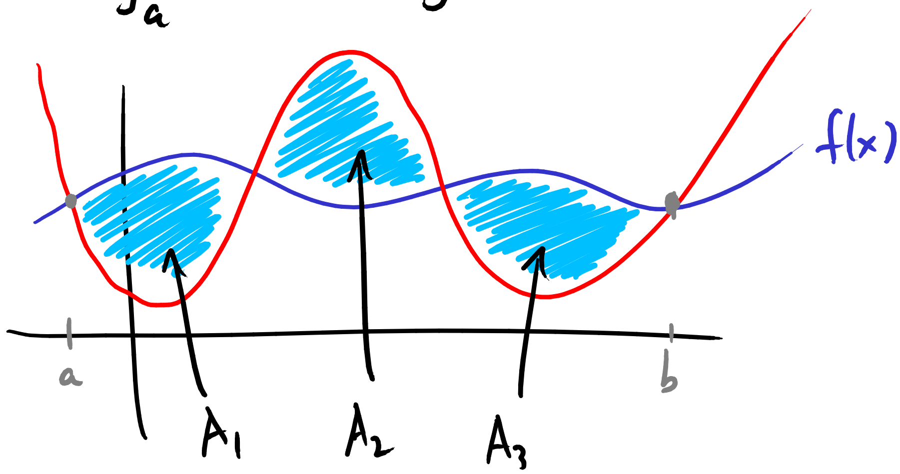
$$= \int_0^1 2x - 2x^2 \, dx$$

$$= x^2 - \frac{2}{3}x^3 \Big|_0^1$$

$$= \underline{\underline{\frac{1}{3}}}$$

A rule that finds the area between $y=f(x)$ and $y=g(x)$ no matter which is bigger:

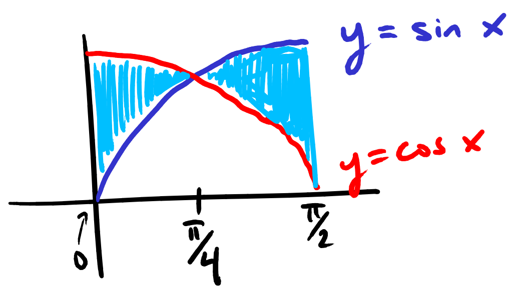
$$A = \int_a^b dx\ |f(x)-g(x)|$$



$$A = \int_a^b |f(x)-g(x)|\,dx = A_1+A_2+A_3$$

To actually calculate this $\int$ of absolute value, usually have to break it up into pieces.

Ex. Find the area of the region between $y=\sin x$ and $y=\cos x$, where x ranges between $x=0$ and $x=\frac{\pi}{2}$.



$$A = \int_0^{\pi/2} |\cos x - \sin x|\,dx = \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$

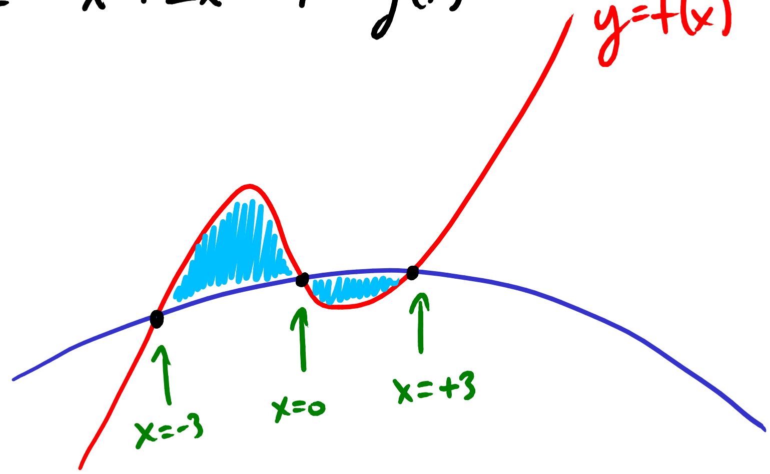
$$= \left(\sin x + \cos x\Big|_0^{\pi/4}\right) + \left(-\cos x - \sin x\Big|_{\pi/4}^{\pi/2}\right)$$

$$= \left(\left(\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}\right) - (0+1)\right) + \left((-0-1) - \left(-\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}\right)\right)$$

$$= \frac{4}{\sqrt{2}} - 2 = \underline{\underline{2\sqrt{2}-2}}$$

Ex Find the area of the region between the curves

$$y = x^3 - x^2 - 7x - 4 = f(x)$$
$$y = -x^2 + 2x - 4 = g(x)$$



First, find the points of <u>intersection</u>:

$$x^3 - x^2 - 7x - 4 = -x^2 + 2x - 4$$
$$x^3 - 9x = 0$$
$$x(x+3)(x-3) = 0$$

$\Longrightarrow$ intersections are $\underline{x = 0, -3, +3}$

Area is $\displaystyle\int_{-3}^{3} |f(x) - g(x)|\, dx$

$$f(x) - g(x) = x^3 - 9x$$

$$= \int_{-3}^{3} |x^3 - 9x|\, dx = \int_{-3}^{3} |x(x+3)(x-3)|\, dx$$

<0	>0	<0	>0
	-3	0	3

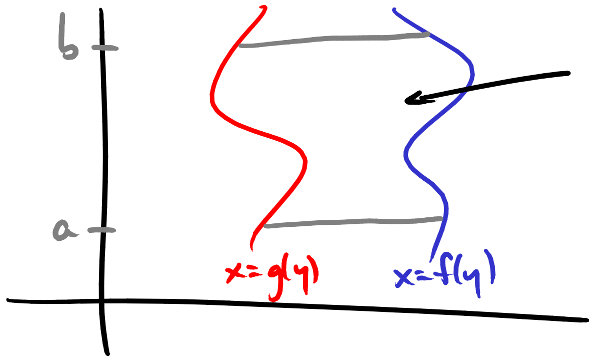
$$= \int_{-3}^{0} x^3 - 9x\, dx + \int_{0}^{3} 9x - x^3\, dx$$

$$= \frac{81}{4} + \frac{81}{4}$$

$$= \frac{81}{2}$$

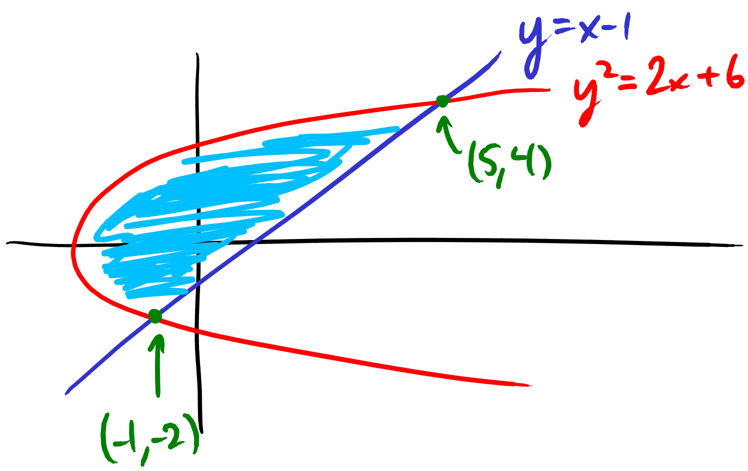
Can also consider two curves $x = f(y)$

$x = g(y)$



$$A = \int_a^b |f(y) - g(y)|\, dy$$

Ex Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



Write the curves as
$$x = y + 1 = f(y)$$
$$x = \tfrac{1}{2}y^2 - 3 = g(y)$$

$$A = \int_{-2}^{4} |f(y) - g(y)|\, dy = \int_{-2}^{4} (y+1) - \left(\tfrac{1}{2}y^2 - 3\right) dy = \underline{\underline{18}}$$