Lecture 9

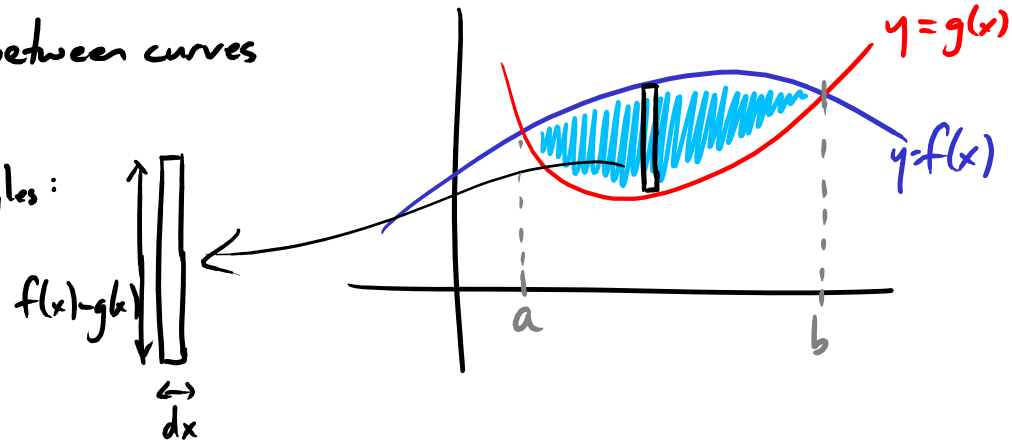
8 Feb 2010

Housekeeping:

1st midterm: 23 Feb 7–9pm WEL 1.316

Last time: Areas between curves

Break into rectangles:

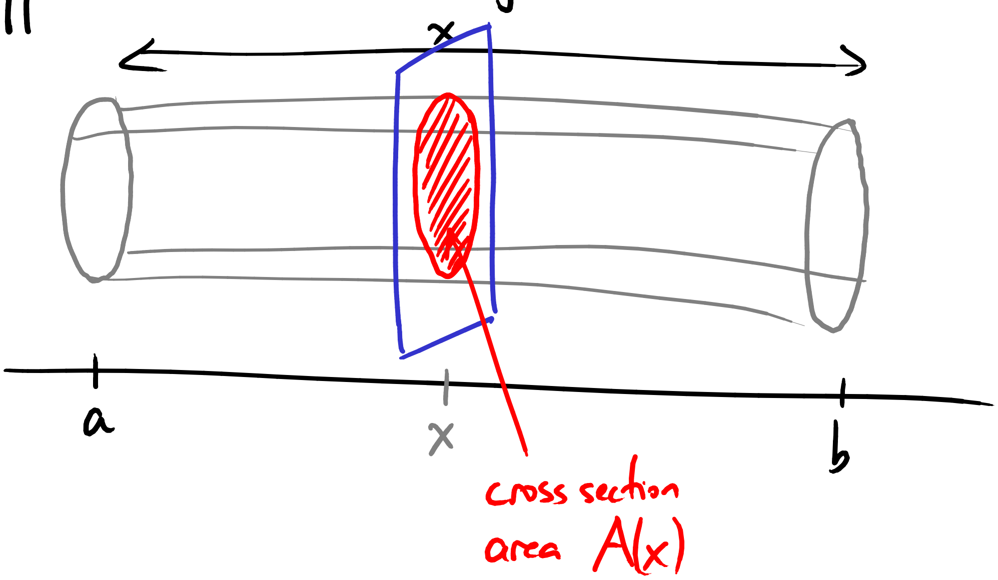


rectangle has $A = (f(x)-g(x))\,dx$

sum them up: total area $= \int_a^b (f(x)-g(x))\,dx$

Volumes (Ch 6.2)

Suppose we have some 3-d object and want to find its volume.

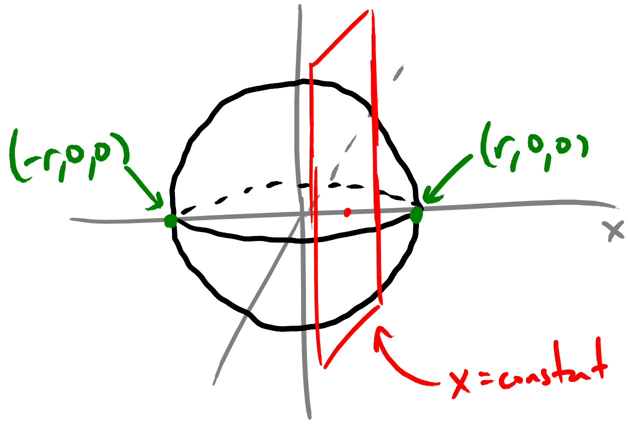


Chop the volume into slices: each slice has volume $A(x)\,dx$

Total volume: add up the slices —

$$V = \int_a^b A(x)\,dx$$

Ex Calculate the volume of a sphere of radius r.



Slice the sphere by planes $x = \text{constant}$.

Sphere is $x^2+y^2+z^2 \le r^2$

At fixed value of x:

this is $\left(y^2+z^2 \le r^2 - x^2\right)$ inside y-z plane.

That's the inside of a circle with radius $\sqrt{r^2-x^2}$.

So the cross sections are circles, with area $\pi(r^2-x^2) = A(x)$.

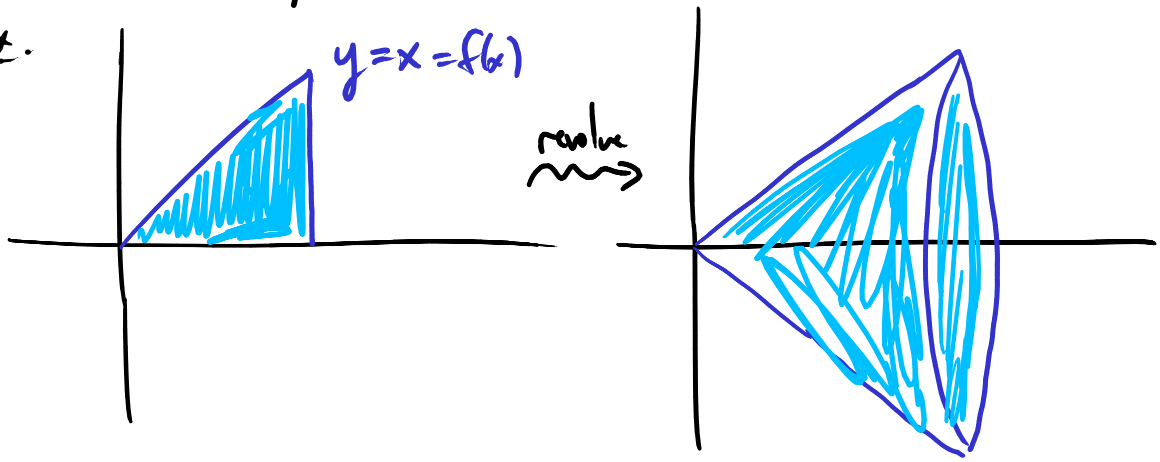
So the volume is $V = \int_{-r}^{r} dx\, A(x) = \int_{-r}^{r} \pi(r^2-x^2)\,dx$

$$= \pi\left(r^2x - \frac{x^3}{3}\right)\Big|_{-r}^{r}$$

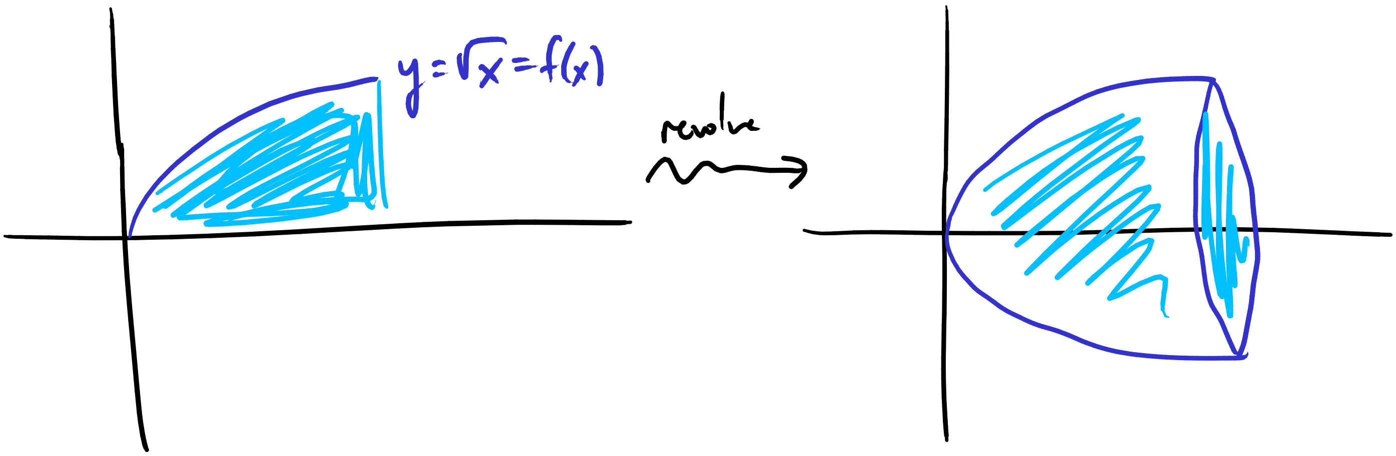
$$= \frac{4}{3}\pi r^3$$

A common class of solid: "solid of revolution" — take the region under some graph and <u>revolve</u> it around, say, the x-axis.

<u>Ex</u>.



<u>cone</u>



The cross-sectional area in this case is just $A(x)=\pi f(x)^2$.
(Because the cross section is a <u>circle</u> with radius = $f(x)$.)

<u>Ex</u> Find the volume of a solid obtained by revolving the area under $y=\sqrt{x}$ around the x-axis, with x from 0 to 2.

$$V=\int_0^2 dx\, A(x)=\int_0^2 dx\, \pi(\sqrt{x})^2=\int_0^2 dx\, \pi x=\underline{\underline{2\pi}}$$

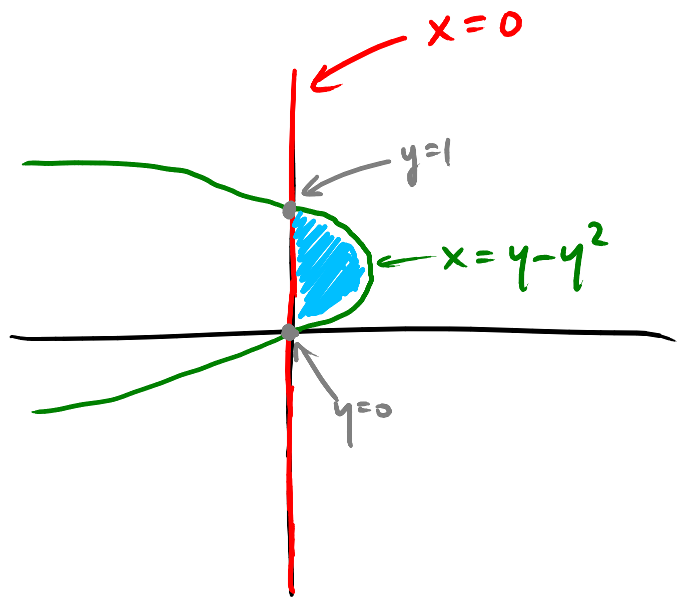
Can also revolve around, say, the y-axis.

<u>Ex</u> Find the volume of a solid obtained by revolving the region between

$$x = y - y^2$$

and $x = 0$

around the y-axis.



Intersections: $y - y^2 = 0$

$y(1-y) = 0$

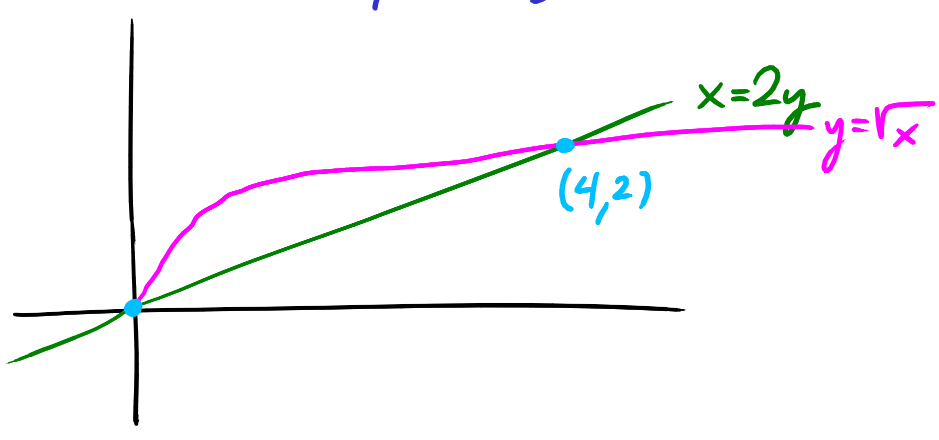
$y = 0,\ y = 1$

Slices at constant y are circles, with <u>radius</u> $y - y^2$

So $V = \int_0^1 dy\, A(y) = \int_0^1 dy\, \pi (y-y^2)^2 = \underline{\underline{\frac{\pi}{30}}}$

We may also encounter solids which can be sliced into little "washers" (circular disc with a circular hole cut out).

<u>Ex</u> Let R be the region between $y = \sqrt{x}$ and $x = 2y$. Find the volume of the solid obtained by rotating R around the y-axis.



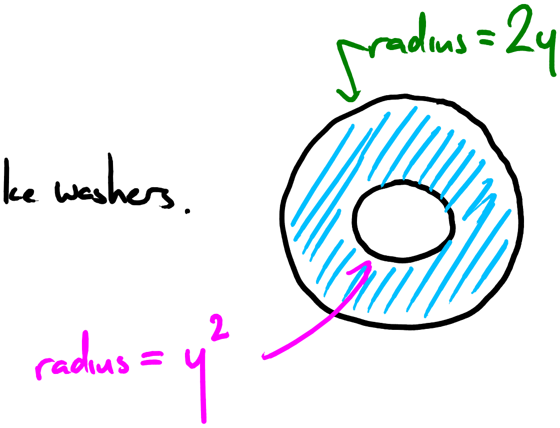
Intersection:

$y = \sqrt{x} \Rightarrow x = y^2$

also $x = 2y$

So $2y = y^2$

$$y^2 - 2y = 0$$
$$y(y-2) = 0 \implies y = 0 \text{ or } y = 2$$



Rotate around y axis: cross sections look like washers.

Radii determined by the distance from the y-axis, i.e. the value of $\underline{\underline{x}}$.

$$\text{Cross section area} = \pi(2y)^2 - \pi(y^2)^2$$

i.e. $A(y) = \pi(4y^2 - y^4)$

$$V = \int_0^2 A(y)\,dy = \int_0^2 \pi(4y^2 - y^4)\,dy = \underline{\underline{\frac{64}{15}\pi}}$$