Lecture 20

15 Mar 2010

Last time: partial fractions

$$\int \frac{P(x)}{Q(x)}\,dx \qquad \text{e.g.} \qquad \int \frac{x^4+4x^3+2x-17}{x^3+2x^2+x}\,dx$$

first do polynomial long division (if deg(numerator) ≥ deg(denom))

then factor the denominator and use that to split up the fraction

Can even use this on simple-looking things:

e.g. $$\int \frac{1}{x(x+1)}\,dx$$

Don't have to do long division here: just write

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Mult. by $x(x+1)$ both sides:

$$1 = A(x+1) + Bx$$
$$1 = (A+B)x + A \Longrightarrow \begin{bmatrix} A=1 \\ A+B=0 \end{bmatrix} \Rightarrow \begin{bmatrix} A=1 \\ B=-1 \end{bmatrix}$$

so $$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

So $$\int \frac{dx}{x(x+1)} = \int dx \left(\frac{1}{x} - \frac{1}{x+1}\right) = \underline{\underline{\ln|x| - \ln|x+1| + C}}$$

If we had instead $\int \frac{1}{x(x^2+1)}$

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Strategy For Integration (Ch 8.5)

1) Basic integration formulas

$\int x^n\,dx = \ldots$

$\int \sin x\,dx = \ldots$

$\int \frac{1}{x}\,dx = \ldots$

(table p. 520 of text)

2) Simplify the integrand (using either algebra or trig identities)

$$\int \sqrt{x}\,(1+\sqrt{x})\,dx = \int (\sqrt{x}+x)\,dx$$

$$\int \frac{\tan\theta}{\sec^2\theta}\,d\theta = \int \frac{\sin\theta}{\cos\theta}\cdot \cos^2\theta\,d\theta = \int \sin\theta\cos\theta\,d\theta = \frac{1}{2}\int \sin 2\theta\,d\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta$

3) "Easy" substitutions

$$\int \frac{x}{x^2-1}\,dx \qquad u = x^2-1,\quad du = 2x\,dx \qquad \ldots$$

4) Classify:

a) Trig $[\sin^a x\cos^b x,\ \tan^a x\sec^b x,\ \cot^a x\csc^b x]$
— use rules from last week

b) Rational function — partial fractions

c) Product of 2 distinct pieces — int. by parts

d) Radicals — $\sqrt{\pm x^2 \pm a^2}$ — trig sub

$\sqrt[n]{ax+b}$ — $u = \sqrt[n]{ax+b}$ sub

5) Try again:

a) look for a clever substitution

b) $\int$ by parts [even if you have to take $u =$ the whole integrand, $dv = dx$

e.g. $\int \tan^{-1} x \, dx$ can be done this way]

c) algebraic manip. [e.g. $\int \frac{dx}{1-\cos x}$: multiply top & bottom by $1+\cos x$]

d) try to relate it to one you've done before

e) combine several methods...

Ex $\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$

[or: use $u=\frac{x}{3}$ and the fact $\int \frac{1}{1+u^2}\,du = \tan^{-1}(u)$]

Ex $\int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \tan^3 x \sec^3 x\,dx$

odd # of powers of tangent $\Rightarrow \int \tan^2 x \sec^2 x\,(\tan x \sec x\,dx)$

$u = \sec x$
$du = \tan x \sec x$

use $\tan^2 x = \sec^2 x - 1$

...

Ex $\int e^{\sqrt{x}}\,dx$ Use $u=\sqrt{x}$ $x=u^2$
$dx = 2u\,du$

$\int = \int e^u (2u\,du)$ Use $\int$ by parts...

Ex $\int \frac{x^5+1}{x^3-3x^2-10x}\,dx$ Partial fractions

Ex $\int \frac{dx}{x\sqrt{\ln x}}$ $u = \ln x$ $\left(du = \frac{dx}{x}\right)$

$= \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du$

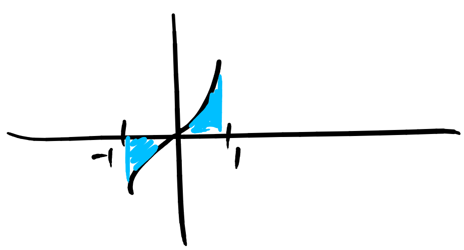
Ex $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\,dx$ mult. top, bottom by $\sqrt{1-x}$

$$\longrightarrow \int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

trig sub: $x = \sin\theta$ [or just $\int = \sin^{-1}x$]

u-sub: $u = 1-x^2$

Ex $\displaystyle\int_{-1}^{1} x^4 \sin x\,dx$ One way: $\int$ by parts (4 times)



Or: use the fact that

$$\int_{-a}^{a} (\text{odd function})\,dx = \underline{\underline{0}}$$

Ex $\displaystyle\int (x+\sin x)^2\,dx$

Multiply out: $\displaystyle\int x^2 + 2x\sin x + \sin^2 x\,dx$

easy by parts $\frac{1}{2}$-angle identity

Ex $\displaystyle\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{dx}{\left(\sqrt{1-x^2}\right)^3}$

$x = \sin\theta$
$dx = \cos\theta\,d\theta$

$$= \int \sec^2\theta\,d\theta = \underline{\underline{\tan\theta + C}}$$