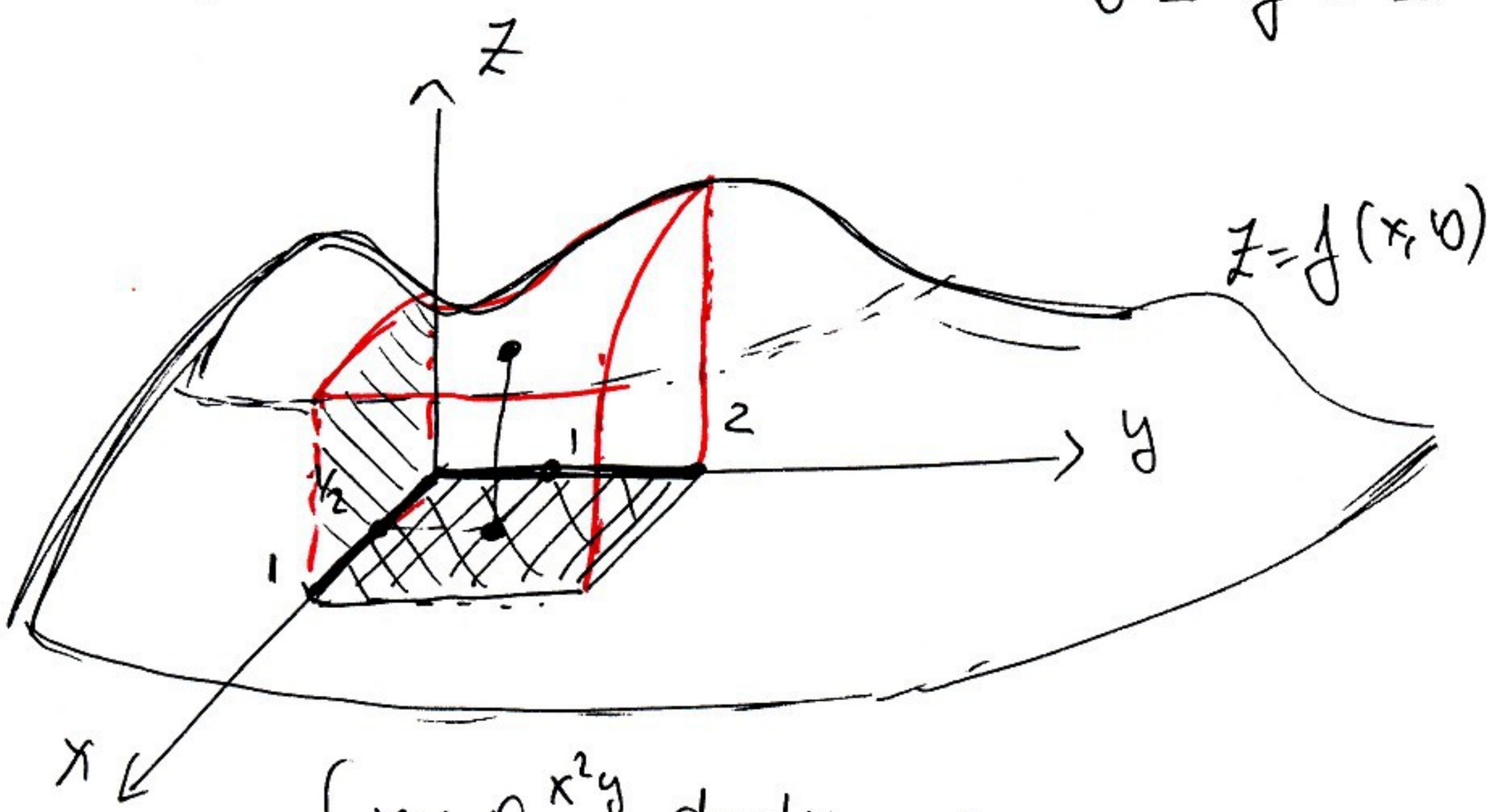
Chapter 16.2 / 16.3

Double Integrals

ex: $z = xy\, e^{x^2 y}$

→ integration domain

$$\mathcal{D} = \left\{ (x, y) \text{ such that } \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 2 \end{array} \right\}$$



$$\iint_{\mathcal{D}} xy\, e^{x^2 y}\, dx\, dy =$$

$$\int_0^2 \underbrace{\left(\int_0^1 xy\, e^{x^2 y}\, dx \right)}_{\text{area}} dy = \int_0^2 y \left(\int_0^1 x\, e^{x^2 y}\, dx \right) dy$$

Compute

$$\int_0^1 x\, e^{x^2 y}\, dx = \int_{2y}^{\perp} u_x = (*)$$

$= \frac{d}{dx} u$

$$\frac{d}{dx}\left(e^{x^2 y}\right) = 2xy \cdot e^{x^2 y}$$

$f' = f_x$

$u = e^{x^2}$

$\left(e^{x^2 y}\right)_x = \frac{d}{dx}\left(e^{x^2 y}\right)$

$\frac{du}{dx} = 2x\, e^{x^2}$

$$(*) = \frac{1}{2y} \int_0^1 \left(e^{x^2 y}\right)_x dx$$

$$= \frac{1}{2y}\left[\, e^{x^2 y} \,\Big|_0^1\, \right] =$$

$$= \frac{1}{2y}\left(e^{y} - 1\right)$$

$$\Rightarrow \int_0^1 x\, e^{x^2 y}\, dx = \frac{1}{2y}\left(e^{y} - 1\right)$$

back to the double integral :

$$\int_0^2 y \left(\underbrace{\int_0^1 x e^{x^2 y} \, dx}_{\qquad\qquad} \right) dy =$$

$$= \int_0^2 y \left(\frac{1}{2y} \left(e^y - 1 \right) \right) dy =$$

$$= \int_0^2 \cancel{y} \cdot \frac{1}{\cancel{2y}} \left(e^y - 1 \right) dy =$$

$$= \int_0^2 \frac{1}{2} \left(e^y - 1 \right) dy = \frac{1}{2} \int_0^2 \left(e^y - 1 \right) dy =$$

$$= \frac{1}{2} \left(e^y - y \right) \Big|_0^2 = \frac{1}{2} \left(e^2 - 2 \right) - \frac{1}{2} \left(1 - 0 \right) =$$

$$= \frac{1}{2} e^2 - 1 - \frac{1}{2} = \boxed{\frac{1}{2} e^2 - \frac{3}{2}}$$

$$\int_0^1 x\, e^{x^2 y}\, dx$$

$u = x^2 y$

$du = 2xy$

$$= \frac{1}{2y} \int_{\text{⊛}}^{\text{⊛}} e^u\, du$$

$$= \frac{1}{2y} \int_{\text{⊛}}^{\text{⊛}} (e^u)_u\, du$$

$$= \int_1^e x^3 \left(y \Big|_0^{\ln(x)} \right) dx$$

$$= \int_1^e x^3 \left(\ln(x) - 0 \right) dx$$

$$= \int_1^e x^3 \ln(x)\, dx \longrightarrow \text{by parts}$$

$$f(x) = \ln(x) \longrightarrow f'(x) = \frac{1}{x}$$

$$g'(x) = x^3 \longrightarrow g(x) = \frac{x^4}{4}$$

$$\int_1^e f'g = fg\Big|_1^e - \int_1^e fg'$$

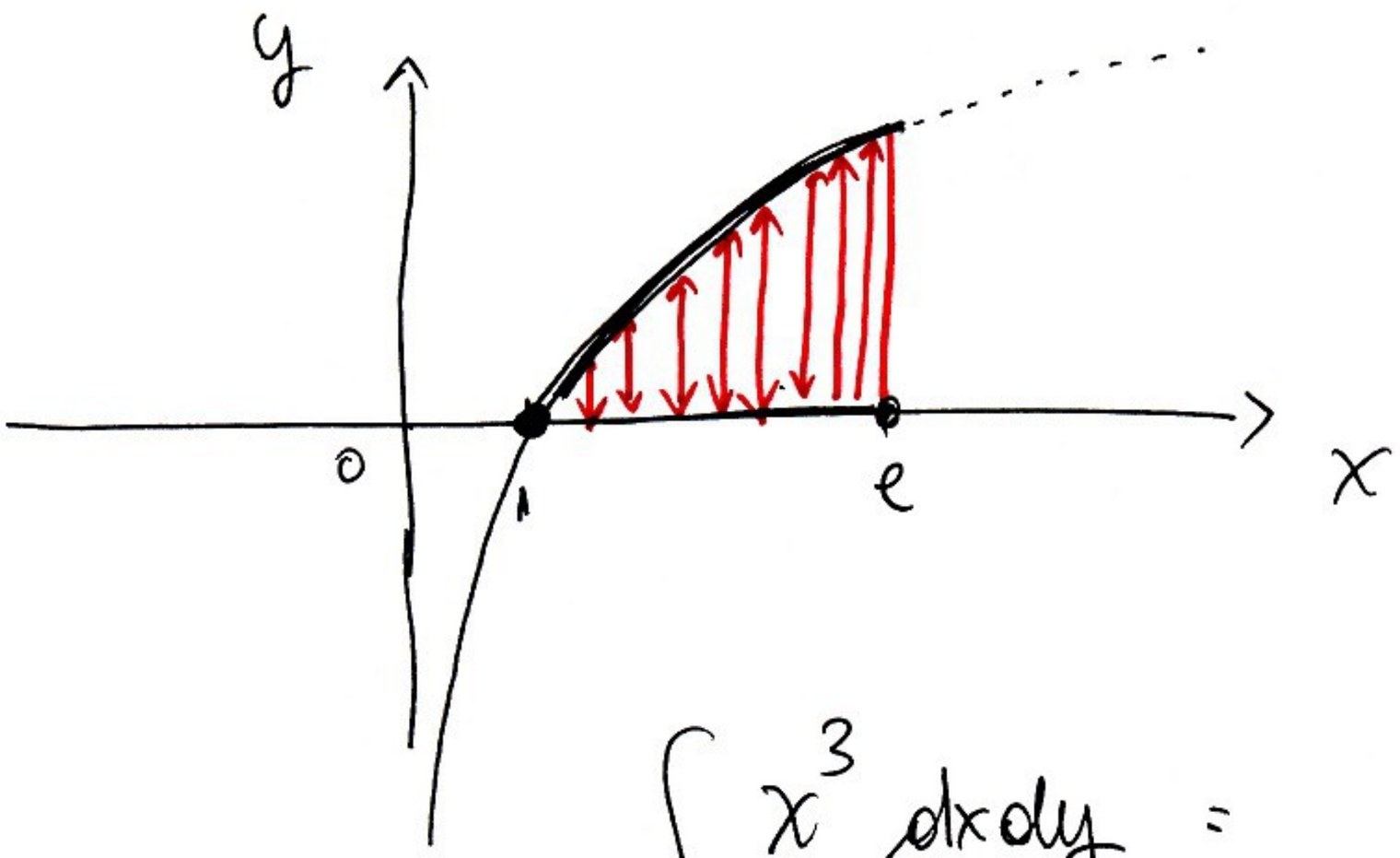
$$\int_1^e x^3 \ln(x)\, dx = \frac{x^4}{4} \cdot \ln(x) \Big|_1^e - \int_1^e \frac{1}{4} x^4 \cdot \frac{1}{x}\, dx$$

$$= \frac{x^4 \cdot \ln(x)}{4} \Big|_1^e - \frac{1}{4} \int_1^e x^3\, dx$$

$$= \frac{e^4 \ln(e)}{4} - \frac{\ln(1)}{4} - \frac{1}{4} \cdot \frac{1}{4} x^4 \Big|_1^e = \dots$$

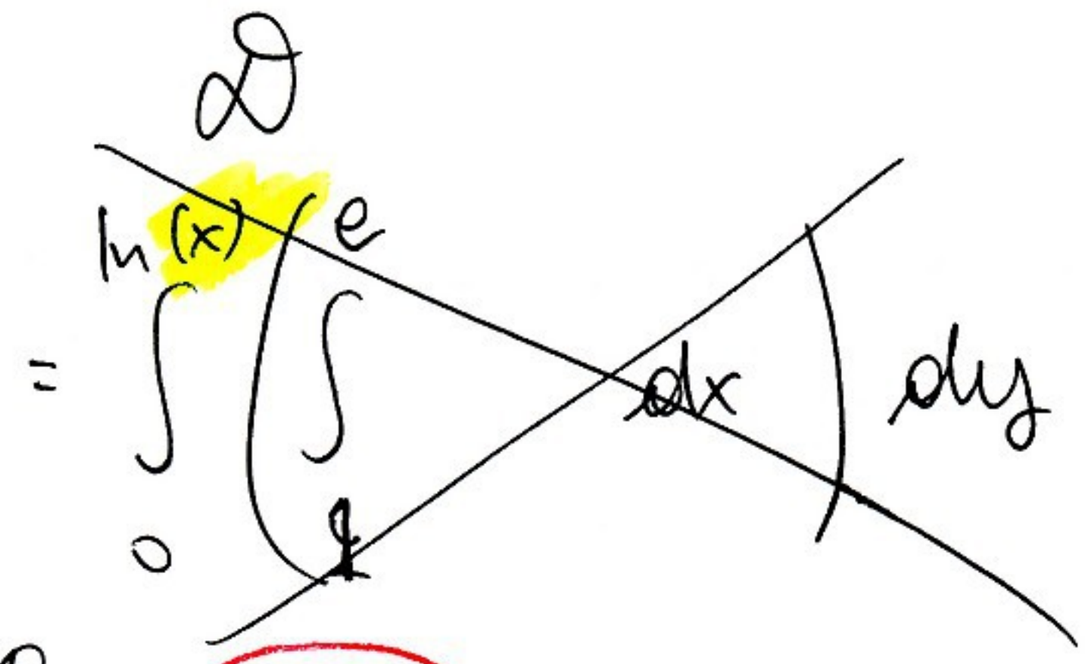
ex: $z = x^3$

$$D = \{ (x,y) \;/\; 1 \le x \le e \,,\; 0 \le y \le \ln(x) \}$$



$$\int x^3 \, dx\,dy =$$

$D \Rightarrow$



$$= \int_1^e \left(\int_0^{\ln(x)} x^3 \, dy \right) dx =$$

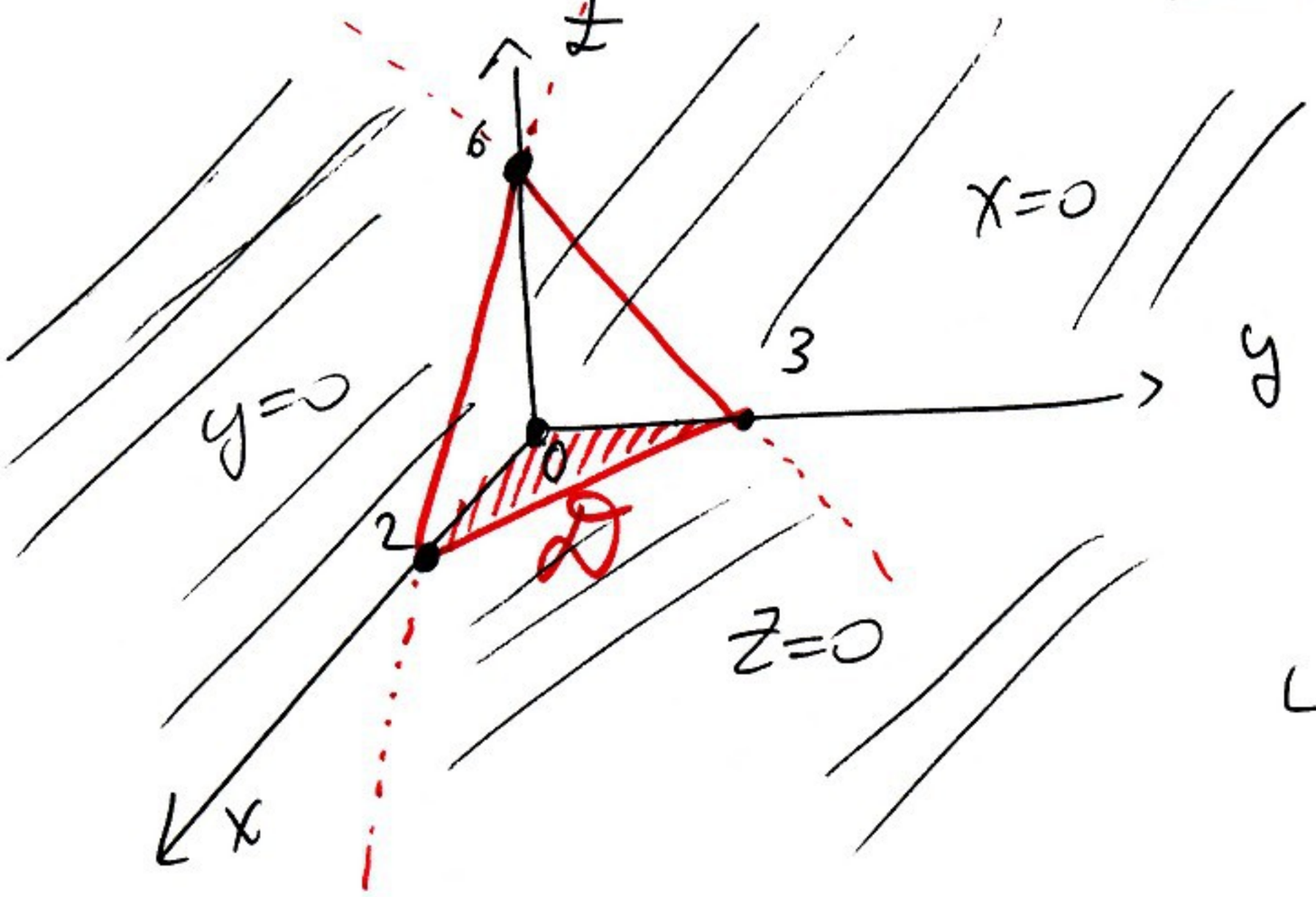
$$= \int_1^e x^3 \left(\int_0^{\ln(x)} dy \right) dx =$$

ex : Volume of a solid $z=0,\ x=0,\ y=0$

bounded by the coordinates planes

and the plane $\boxed{3x+2y+z=6}$

$z = 6-3x-2y$



$$\iint_{D} 6-3x-2y \, dx\,dy$$

D

Volume

$$D = \{ (x,y) \text{ such that}$$

$$\leq x \leq$$

$$\leq y \leq$$