Lecture 31 7 Apr 2010

Last time: sequences. a_n

Recall summation notation e.g. $1+4+9+16+25$

$$= 1^2+2^2+3^2+4^2+5^2$$

$$= \sum_{i=1}^{5} i^2$$

Series (or Infinite Series) (Ch 12.2)

Take a sequence a_n. Try to take sum of all the terms of the seq.

Ex $a_i = i$: $1+2+3+4+5+\cdots = \sum_{i=1}^{\infty} i$

Ex $a_i = \frac{1}{2^i}$: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots = \sum_{i=1}^{\infty} \frac{1}{2^i}$

What do these sums mean?
Like an improper integral $\int^{\infty}$: look at the sum of n terms,

$$S_n = \sum_{i=1}^{n} a_i \quad (\text{"partial sum"})$$

Then try to take the limit as $n \to \infty$. If it exists, we say the series converges, otherwise it diverges.

Ex if $a_i = i$:

$$S_1 = 1$$
$$S_2 = 1+2 = 3$$
$$S_3 = 1+2+3 = 6$$
$$S_4 = 1+2+3+4 = 10$$

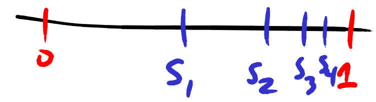
The s_n don't converge (i.e. $\lim_{n\to\infty} s_n$ doesn't exist)

So $\sum_{i=1}^{\infty} i$ doesn't converge (diverges).

Ex if $a_i = \frac{1}{2^i}$:

$$s_1 = \frac{1}{2}$$

$$s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$



$$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$s_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

$$\vdots$$

$\lim_{n\to\infty} s_n = 1$. So we say $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$ (converges to 1).

[NB: $s_n - s_{n-1} = a_n$]

Basic example: <u>geometric series</u> e.g. $2, 6, 18, 54, \ldots$ $\begin{bmatrix} a=2 \\ r=3 \end{bmatrix}$

$2 \xrightarrow{\times 3} 6 \xrightarrow{\times 3} 18 \xrightarrow{\times 3} 54$

$a_n = a \cdot r^{n-1}$

a: 1^{st} term

r: ratio of successive terms

What's $\sum_{i=1}^{\infty} a \cdot r^{i-1}$?

Look at the partial sums:

$$s_n = \sum_{i=1}^{n} a \cdot r^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1}$$

subtract 2 equations:

$$\begin{array}{rl} s_n = & a + ar + ar^2 + \cdots + ar^{n-1} \\ - \quad r s_n = & \phantom{a +{}} ar + ar^2 + \cdots + ar^{n-1} + ar^n \\ \hline s_n - r s_n = & a - ar^n \end{array}$$

$$s_n(1-r) = a(1-r^n)$$

So: $s_n = a \dfrac{1-r^n}{1-r}$ (if $r \neq 1$)

We're interested in $\lim_{n\to\infty} s_n$. If $|r|<1$ then $\lim_{n\to\infty} r^n = 0$

so $\lim_{n\to\infty} s_n = \dfrac{a}{1-r}$

If $|r|>1$ or $r=-1$ then $\lim_{n\to\infty} s_n$ doesn't exist (b/c $\lim_{n\to\infty} r^n$ doesn't)

Also if $r=1$ $\lim_{n\to\infty}$ doesn't exist

Summary: $$\sum_{i=1}^{\infty} a \cdot r^{i-1} = \begin{cases} \dfrac{a}{1-r} & \text{if } |r|<1 \\ \text{divergent} & \text{if } |r| \geq 1 \end{cases}$$

Ex Find the sum of the series

$$2 + \frac{1}{3} + \frac{1}{18} + \frac{1}{108} + \frac{1}{648} + \cdots$$

$\times\frac{1}{6}$ $\times\frac{1}{6}$ $\times\frac{1}{6}$ $\times\frac{1}{6}$

Geometric: $a = 2$

$r = \frac{1}{6}$

$$\text{sum} = \frac{a}{1-r} = \frac{2}{1-\frac{1}{6}} = \frac{2}{\left(\frac{5}{6}\right)} = \frac{12}{5}$$

Ex $4 - 3 + \frac{9}{4} - \frac{27}{16} + \cdots$

$\times -\frac{3}{4}$ $\times -\frac{3}{4}$ $\times -\frac{3}{4}$

Geometric: $a = 4$

$r = -\frac{3}{4}$

$\left|-\frac{3}{4}\right| < 1 \Rightarrow$ converges

$$\text{sum} = \frac{a}{1-r} = \frac{4}{1-\left(-\frac{3}{4}\right)} = \frac{4}{\left(\frac{7}{4}\right)} = \underline{\underline{\frac{16}{7}}}$$

Ex Does $\displaystyle\sum_{i=1}^{\infty} \frac{10^i}{(-9)^{i-1}}$ converge?

First: is it geometric? <u>Yes</u>. 2 ways to see this:

<u>METHOD 1</u>

$$10 \cdot \frac{10^{i-1}}{(-9)^{i-1}} = 10 \cdot \left(-\frac{10}{9}\right)^{i-1} = ar^{i-1}$$

so our series is $\displaystyle\sum_{i=1}^{\infty} ar^{i-1}$ with $a = 10$ and $r = -\frac{10}{9}$

METHOD 2

Divide $\dfrac{a_{i+1}}{a_i} = \dfrac{10^{i+1}/(-9)^i}{10^i/(-9)^{i-1}} = \dfrac{10}{-9}$

Since this ratio is constant, we have a geometric series, with $r = -\frac{10}{9}$

and $a = a_1 = \dfrac{10^1}{(-9)^{1-1}} = 10$

Since the series is geometric, with $|r| = \left|-\frac{10}{9}\right| = \frac{10}{9} > 1$,

the series diverges.

Ex Calculate the sum $\displaystyle\sum_{i=1}^{\infty} \frac{3+5^i}{7^i}$.

This is not geometric, but it is the sum of two geometric series:

$$\sum_{i=1}^{\infty} \frac{3+5^i}{7^i} = \sum_{i=1}^{\infty} \frac{3}{7^i} + \sum_{i=1}^{\infty} \left(\frac{5}{7}\right)^i$$

geom. series with $a = \frac{3}{7}$, $r = \frac{1}{7}$

geom. with $a = \frac{5}{7}$, $r = \frac{5}{7}$

$\Rightarrow$ converges to $\dfrac{a}{1-r} = \dfrac{\frac{3}{7}}{1-\frac{1}{7}} = \dfrac{1}{2}$

$\Rightarrow$ converges to $\dfrac{\frac{5}{7}}{1-\frac{5}{7}} = \dfrac{5}{2}$

$$= \frac{1}{2} + \frac{5}{2} = 3$$