Lecture 33

12 Apr 2010

Last lecture: a few ways of seeing whether a series converges.

- If it's a geometric series, look at the common ratio r.
  - If $|r| \geq 1$, series diverges
  - If $|r| < 1$, series converges

How to tell whether the series is geometric?
Look at the ratios of successive terms: if they're all the same then the series is geometric.

Ex $\quad 2+4+6+8+10+\cdots$

$\frac{4}{2}=2 \quad \frac{6}{4}=\frac{3}{2} \quad \frac{8}{6}=\frac{4}{3} \quad \cdots$

these ratios aren't equal $\Rightarrow$ the series is not geometric.

Ex $\quad 1+2+4+8+16+\cdots$

$\frac{2}{1}=2 \quad \frac{4}{2}=2 \quad \frac{8}{4}=2 \quad \cdots$

the ratios are all 2 $\Rightarrow$ the series is geometric, with $r=2$.

Since $2>1$, the series doesn't converge (diverges).

Another test (Test For Divergence):

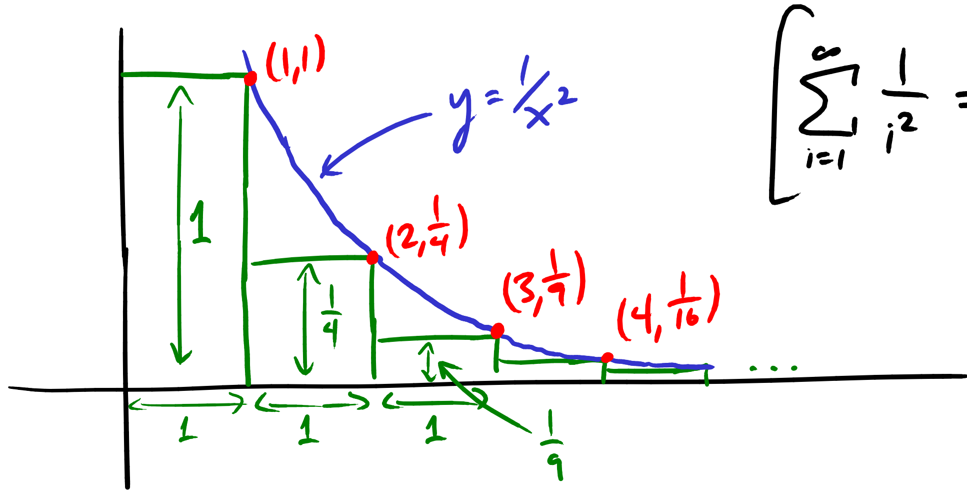
If $\lim_{i\to\infty} a_i$ doesn't exist, or if $\lim_{i\to\infty} a_i$ does exist but $\lim_{i\to\infty} a_i \neq 0$, then $\sum_{i=1}^{\infty} a_i$ diverges.

Integral Test (Ch 12.3)

Take the series $\sum_{i=1}^{\infty} \frac{1}{i^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$

Not geometric. And $\lim_{i\to\infty} \frac{1}{i^2} = 0$. So it *might* converge.

To see for sure:



$\left[\sum_{i=1}^{\infty} \frac{1}{i^2} = \text{sum of areas of these rectangles}\right]$

$A = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

The picture shows that $\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ is <u>less than</u> $\int_1^{\infty} \frac{1}{x^2}\,dx$.

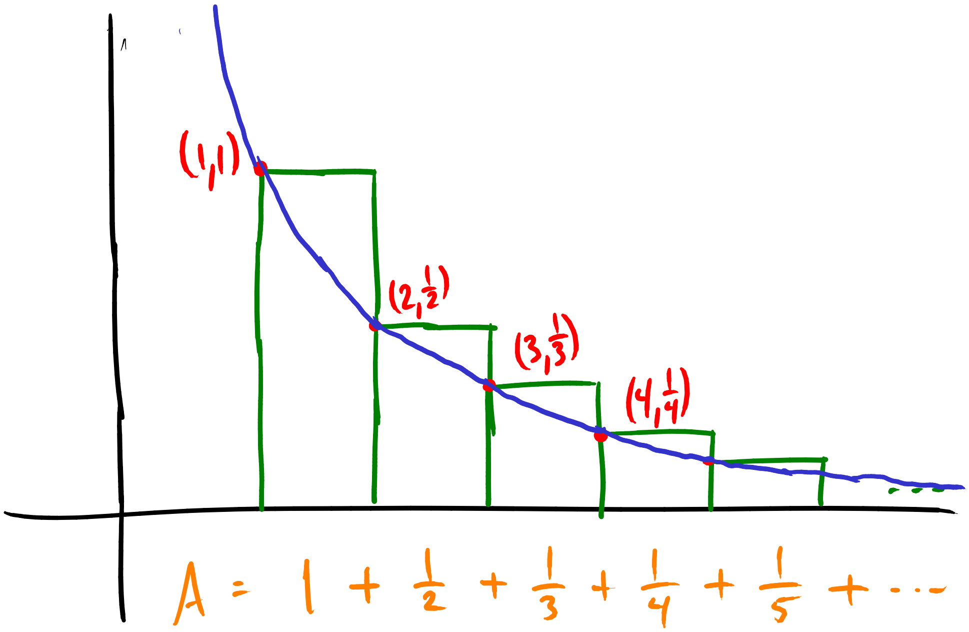
But $\int_1^{\infty} \frac{1}{x^2}\,dx$ <u>converges</u> (see lecture on improper integrals)

So $\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ must <u>also converge</u>!

So: $\sum_{i=1}^{\infty} \frac{1}{i^2}$ <u>converges</u>.

(Adding the 1 in front to get $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ doesn't affect convergence.)

How about $\sum_{i=1}^{\infty} \frac{1}{i}$?



$A = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

The picture shows that $\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right) > \int_1^\infty \frac{1}{x}\,dx.$

But $\int_1^\infty \frac{1}{x}\,dx$ <u>diverges</u> (cf. previous lecture on improper $\int$)

$$\left[\text{because } \int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln(t) = \infty\right]$$

So $\sum_{i=1}^\infty \frac{1}{i}$ <u>diverges</u>!

General rule (Integral Test):

Suppose $f(x)$ is a <u>continuous</u> <u>decreasing</u> function, defined for $1 \le x < \infty$. Say $a_i = f(i)$.

Then:

If $\int_1^\infty f(x)\,dx$ is <u>convergent</u> then $\sum_{i=1}^\infty a_i$ is <u>convergent</u>

If $\int_1^\infty f(x)\,dx$ is <u>divergent</u> then $\sum_{i=1}^\infty a_i$ is <u>divergent</u>

Ex Does $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^2+1}$ converge or diverge?

$f(x) = \dfrac{1}{x^2+1}$ is decreasing

so look at $\displaystyle\int_1^{\infty} \frac{1}{x^2+1}\,dx$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \left(\tan^{-1} x \Big|_1^t \right) = \lim_{t\to\infty} \left(\tan^{-1} t - \frac{\pi}{4} \right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}, \text{ converges}$$

So $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^2+1}$ converges, by the Integral Test.

Example For what values of p is $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^p}$ convergent?

("p-series")

If $p<0$: $\displaystyle\lim_{i\to\infty} \frac{1}{i^p} = \lim_{i\to\infty} i^{-p} = \infty$ so diverges

If $p=0$: $\displaystyle\lim_{i\to\infty} \frac{1}{i^p} = 1$ so diverges

If $p>0$: $f(x) = \dfrac{1}{x^p}$ is decreasing so use Integral Test:

look at $\int_1^\infty \frac{1}{x^p}\,dx$. We know this $\begin{cases} \underline{\text{converges}} \text{ if } p>1 \\ \underline{\text{diverges}} \text{ if } p\le 1 \end{cases}$

Summarizing (p-test):

$$\sum_{i=1}^{\infty} \frac{1}{i^p} \begin{cases} \underline{\text{converges}} \text{ if } p>1 \\ \underline{\text{diverges}} \text{ if } p\le 1 \end{cases}$$