Lecture 34 | 14 Apr 2010

Last time: "Integral Test":

If $f(x)$ is continuous, decreasing for $x > M$ (M = any #)

and $a_n = f(n)$

then $\sum_{n=M}^{\infty} a_n$ $\begin{cases} \text{converges if } \int_M^\infty f(x)\,dx \text{ converges} \\ \text{diverges if } \int_M^\infty f(x)\,dx \text{ diverges} \end{cases}$

"p-test":

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{converges if } p>1 \\ \text{diverges if } p \leq 1 \end{cases}$

Ex Does $\sum_{k=31}^{\infty} k e^{-k}$ converge?

Try to use Integral Test: look at $f(x) = xe^{-x}$

Is $f(x)$ decreasing? $f'(x) = e^{-x} + x\cdot(-e^{-x})$

$$= \underbrace{(1-x)}_{\text{negative if } x>1}\ \underbrace{e^{-x}}_{\text{positive}}$$

So if $x>1$, $f'(x) < 0$ i.e. $f(x)$ is decreasing if $x > 1$.

So we can apply Integral Test:

Look at $\int_1^\infty xe^{-x}\,dx = \lim_{t\to\infty} \int_1^t xe^{-x}\,dx$

by Int. By Parts, can see that limit exists

so $\int_1^\infty x e^{-x}\,dx$ converges

so $\sum_{k=1}^{\infty} k e^{-k}$ converges by Int. Test

Comparison Tests (Ch 12.4)

Comparison Test:

Suppose we have two sequences of positive numbers: a_n, b_n

1) If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$
then $\sum_{n=1}^{\infty} a_n$ is convergent.

2) If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$
then $\sum_{n=1}^{\infty} a_n$ is divergent.

Ex $\sum_{n=1}^{\infty} \frac{5}{2n^3+4n+3}$: Say $a_n = \frac{5}{2n^3+4n+3}$, $b_n = \frac{5}{2n^3}$

$$\frac{5}{2n^3+4n+3} < \frac{5}{2n^3}$$

and $\sum_{n=1}^{\infty} \frac{5}{2n^3}$ converges by p-test $(p=3>1)$

So $\sum \frac{5}{2n^3+4n+3}$ converges by Comparison Test

Ex $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Comparison: $\frac{\ln n}{n} > \frac{1}{n}$ (when $n>3$)

And $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (by p-test, with $p=1$)

So $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by Comparison Test.

Limit Comparison Test:

Suppose a_n, b_n are two sequences of positive numbers and $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ with $c \neq 0$

Then: if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

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Ex Does $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converge? Say $a_n = \frac{1}{2^n-1}$, $b_n = \frac{1}{2^n}$

- $\lim_{n\to\infty} \frac{\left(\frac{1}{2^n-1}\right)}{\left(\frac{1}{2^n}\right)} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-2^{-n}} = 1$

- $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$. Geometric series with $r = \frac{1}{2}$.

  Since $\left|\frac{1}{2}\right| < 1$, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

So: by Limit Comparison Test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges.

Ex Does $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{n^5 + 5}}$ converge?

For large n, $a_n = \frac{2n^2+3n}{\sqrt{n^5+5}} \sim \frac{2n^2}{\sqrt{n^5}} = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$

So let's try using Lim-Comp Test with $b_n = \frac{2}{n^{1/2}}$

- $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{2n^2+3n}{\sqrt{n^5+5}}\right)}{\left(\frac{2}{n^{1/2}}\right)} = \lim_{n\to\infty} \frac{\left[\frac{n^2\left(2+\frac{3}{n}\right)}{n^{5/2}\sqrt{1+5n^{-5/2}}}\right]}{\left(\frac{2}{n^{1/2}}\right)}$$

$$= \lim_{n\to\infty} \frac{2+\overset{\nearrow 0}{\frac{3}{n}}}{2\sqrt{1+\underset{\searrow 0}{5n^{-5/2}}}} = \lim_{n\to\infty} \frac{2}{2} = 1$$

- $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{n^{1/2}}$ diverges by p-test $(p = \frac{1}{2} < 1)$

So: by Lim-Comp Test, $\sum_{n=1}^{\infty} a_n$ diverges.

Ex Does $\sum_{n=1}^{\infty} \frac{n+1}{(n)4^n}$ converge?

Use Limit-Comparison Test with

$$a_n = \frac{n+1}{n4^n}, \quad b_n = \frac{1}{4^n}$$

- $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{n4^n} \cdot 4^n = \lim_{n\to\infty} \frac{n+1}{n} = 1$
- And $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{4^n}$ converges (geometric, $r = \frac{1}{4}$)

So: $\sum_{n=1}^{\infty} a_n$ converges.