Lecture 39 | 26 Apr 2010

Housekeeping:

- UT Learning Center exam review tonight and tomorrow 6-8pm
- Course Instructor surveys now open at
  https://utdirect.utexas.edu/diia/ecis/
- Exam 3 next Tuesday 7-9pm

A quick comment about sequences vs series:

Take the sequence $a_n = \frac{1}{n}$

We may ask 2 different questions about this sequence.

Q1) Does the sequence $\{a_n\}$ converge?

A1) Yes: $\lim_{n\to\infty} \frac{1}{n} = 0$.

Q2) Does the series $\sum_{n=1}^{\infty} a_n$ converge?

A2) No: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test.

Last time: Power series $\sum_{n=0}^{\infty} c_n (x-a)^n$

Functions As Power Series (Ch 12.9)

Remember the formula $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ (geometric series first term $= 1$ common ratio $= x$)

(for $|x| < 1$)

Ex: $\dfrac{1}{0.7} = \dfrac{1}{1-0.3} = \sum_{n=0}^{\infty} (0.3)^n$

$$= 1 + 0.3 + (0.3)^2 + (0.3)^3 + \cdots$$
$$= 1 + 0.3 + 0.09 + \cdots$$
$$\approx 1.4$$

Ex Find a representation of the function $\dfrac{1}{1+x^2}$ as a power series, and its interval of convergence, radius of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n (x^2)^n$$
$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

To find interval of convergence: could use Ratio Test.
But here there's shortcut: this is a geometric series with common ratio $r = -x^2$.

So it converges if and only if $|r| < 1$

i.e. $|-x^2| < 1$

i.e. $|x|^2 < 1$

$|x| < 1$

So the interval of convergence is $\underline{(-1, 1)}$. Radius of conv. $\underline{\underline{R = 1}}$.

Ex Write $\frac{1}{x+7}$ as a power series.

$$\frac{1}{x+7} = \frac{1}{7}\left(\frac{1}{\frac{x}{7}+1}\right)$$

$$= \frac{1}{7}\left(\frac{1}{1-\left(-\frac{x}{7}\right)}\right)$$

$$= \frac{1}{7}\left(\sum_{n=0}^{\infty}\left(-\frac{x}{7}\right)^n\right)$$

$$= \frac{1}{7}\sum_{n=0}^{\infty}\frac{(-1)^n}{7^n}x^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{7^{n+1}}x^n$$

[power series centered at 0]

[We might also have written $\frac{1}{x+7} = \frac{1}{1-(-6-x)}$

and then get $\frac{1}{x+7} = \sum_{n=0}^{\infty}(-6-x)^n = \sum_{n=0}^{\infty}(-1)^n(x+6)^n$

That's another power series for the same function, centered at -6.]

Ex Write $\frac{x^4}{x+7}$ as a power series (centered at 0).

$$\frac{x^4}{x+7} = x^4\cdot\frac{1}{x+7} = x^4\cdot\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n}{7^{n+1}}x^n}$$

from the previous example

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{7^{n+1}}x^{n+4}$$

Could also rewrite this: let $m = n+4$ (so $n = m-4$)

then the above also equals

$$= \sum_{m=4}^{\infty} \frac{(-1)^{m-4}}{7^{m-3}} x^m$$

$(-1)^{m-4} = (-1)^m (-1)^{-4}$

$$= \sum_{m=4}^{\infty} \frac{(-1)^m}{7^{m-3}} x^m$$

<u>Fact</u>: If we have a power series for $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

Then also:

$$\frac{d}{dx} f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=1}^{\infty} c_n \cdot n (x-a)^{n-1}$$

$$\int f(x)\, dx = \int \sum_{n=0}^{\infty} c_n (x-a)^n \, dx = \left(\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \right) + C$$

Both of these new series have the <u>same</u> radius of convergence as the original one.

<u>Ex</u> Express $\frac{1}{(1-x)^2}$ as a power series, find its radius of conv. (centered at $x=0$)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (\text{radius of conv} = 1)$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx} \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$ (also with radius of conv $= 1$)

<u>Ex</u> Express $\ln(1-x)$ as a power series (centered at 0).

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\int \frac{1}{1-x}\,dx = \int \sum_{n=0}^{\infty} x^n\,dx$$

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} - C$$

To determine C: plug in $x=0$, then the eq. becomes

$$\ln(1) = 0 - C$$

$$0 = 0 - C \qquad \text{so} \qquad C = 0$$

so we get

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

[Also can rewrite this: set $m = n+1$, then

$$\ln(1-x) = -\sum_{m=1}^{\infty} \frac{x^m}{m}$$]