Lecture 42 | 3 May 2010

Exam 3 tomorrow 7–9pm NO CALCULATORS
WEL 2.246

18 questions:
all series
8 power series (incl. Taylor series, Taylor poly...)
7–8 tests for convergence of non-power series
2–3 other Q on non-power series

THINGS WORTH KNOWING:

Tests for convergence —

- Remember the "easy" examples

$$\sum_{n=0}^{\infty} \frac{P(n)}{Q(n)} \qquad P, Q \text{ polynomials}$$

Can always do them by lim-comp and p-test.
The answer always just depends on the leading powers of n.

Ex $$\sum_{n=0}^{\infty} \frac{n^2 - 3n + 9}{n^{5/2} + 4} \quad \text{lim-comp to} \quad \sum_{n=0}^{\infty} \frac{n^2}{n^{5/2}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$$

which diverges by p-test ($p = \frac{1}{2} < 1$).

Ex $$\sum_{n=0}^{\infty} \frac{n^4 - 7n}{n^6 + 8n^5} \quad \text{lim-comp to} \quad \sum_{n=0}^{\infty} \frac{n^4}{n^6} = \sum \frac{1}{n^2}$$

which converges by p-test $(p=2>1)$.

- Always pay attention to what the Q asks:

  "Which of these series converges"

  "Which of these series diverges"

  "Which of them " converges conditionally"

- Most important tests are Ratio Test, p-test, Test For Divergence.

  Test For Divergence: if $\lim_{n\to\infty} a_n \neq 0$ (or doesn't exist)

  then $\sum_{n=0}^{\infty} a_n$ diverges.

  Ex $\sum_{n=0}^{\infty} \frac{n}{\ln n}(-1)^n$ diverges by Test For Divergence

  (in fact $\lim_{n\to\infty} \frac{n}{\ln n} = \infty$

  so $\lim_{n\to\infty} \frac{n}{\ln n}(-1)^n$ doesn't exist)

- (But also remember other tests.)

- Root Test:

  Ex $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}+1}\right)^{3n}$

$$\sqrt[n]{\left|\frac{1}{\sqrt{n}+1}\right|^{3n}} = \left[\left(\frac{1}{\sqrt{n}+1}\right)^{3n}\right]^{1/n} = \left(\frac{1}{\sqrt{n}+1}\right)^{3n\cdot\frac{1}{n}}$$

$$= \left(\frac{1}{\sqrt{n}+1}\right)^{3} \xrightarrow[n\to\infty]{} \underset{L}{\overset{}{0}}$$

Since $L = 0 < 1$, the $\sum$ <u>converges</u> (absolutely).

[If we had gotten $L > 1$, then $\sum$ <u>diverges</u>.
If " " " $L = 1$, then we get <u>no information</u> (test inconclusive).]

Simplifying factorials in the Ratio Test:

<u>Ex</u> $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$

Ratio Test: $$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{(n+1)!}{(2n+3)!}}{\frac{n!}{(2n+1)!}} = \frac{(2n+1)!}{(2n+3)!}\cdot\frac{(n+1)!}{n!}$$

Use $(n+1)! = (n+1)n!$
$(2n+3)! = (2n+3)(2n+2)(2n+1)!$

so the ratio simplifies to $$\frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!}\quad\frac{(n+1)n!}{n!}$$

$$= \frac{n+1}{(2n+3)(2n+2)} \longrightarrow 0 \text{ as } n\to\infty$$

Since $0 < 1$, the series <u>converges</u> (absolutely).

- If Q asks whether a series is
  - absolutely conv.
  - conditionally conv.
  - divergent

check absolute convergence first!

Ex $\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n^3+4}$

Check absolute conv:

$$\sum_{n=1}^{\infty} \left|(-1)^n \frac{n+3}{n^3+4}\right| = \sum_{n=1}^{\infty} \frac{n+3}{n^3+4}$$

which can be done by lim-comp to $\sum \frac{n}{n^3} = \sum \frac{1}{n^2}$

which converges by p-test $(p=2)$

So $\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n^3+4}$ converges absolutely.

Ex $\sum \frac{3^n}{n^4+7n}$ div by Test For Div

$\sum \frac{n^2+6n}{3^n}$ conv by Ratio Test

$$\left[\frac{|a_{n+1}|}{|a_n|} = \frac{\left(\frac{(n+1)^2+6(n+1)}{3^{n+1}}\right)}{\frac{n^2+6n}{3^n}} \xrightarrow{n\to\infty} \frac{1}{3} < 1 \right]$$

- Remember that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

$\left(\text{also } \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{bn} = e^{ab}\right)$.

Power series:

Power series centered at a: $\sum_{n=0}^{\infty} c_n (x-a)^n$

Remember interval of convergence and radius of convergence:

Ex $\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n n^3$ What are int. and rad. of conv.?

Use Ratio Test: $\frac{|a_{n+1}|}{|a_n|} = \frac{\left|\frac{x}{4}\right|^{n+1}(n+1)^3}{\left|\frac{x}{4}\right|^n (n^3)} = \left|\frac{x}{4}\right|\left(\frac{n+1}{n}\right)^3 \to \left|\frac{x}{4}\right|$ as $n\to\infty$

So the series converges if $\left|\frac{x}{4}\right| < 1$ i.e. $|x| < 4$

diverges if $\left|\frac{x}{4}\right| > 1$ i.e. $|x| > 4$

So radius of convergence $= 4$

Interval of convergence: is it $[-4,4]$ or $(-4,4]$

or $[-4,4)$ or $(-4,4)$?

Check endpoints: plug in $x=4$ $\sum_{n=0}^{\infty} (1)^n n^3$ diverges by TFD

plug in $x=-4$ $\sum_{n=0}^{\infty} (-1)^n n^3$ diverges by TFD

So int. of conv. is $(-4,4)$.

Remember the Taylor series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ex Find Taylor series centered at $a=0$ for

$x e^{2x^3}$

Use the series for e^x: $e^{2x^3} = \sum_{n=0}^{\infty} \frac{(2x^3)^n}{n!}$

$$x e^{2x^3} = x \sum_{n=0}^{\infty} \frac{(2x^3)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{2^n x^{3n+1}}{n!}$$

Ex Find the Taylor polynomial of deg $=2$ around $a=4$ for $f(x)=\sqrt{x}$.

$$T_2(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} \quad \ldots$$

$f(x) = \sqrt{x}$ $\qquad$ $f(a) = \sqrt{4} = 2$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\qquad$ $f'(a) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$$f''(x) = -\frac{1}{4x^{3/2}} \qquad f''(a) = -\frac{1}{4 \cdot 4^{3/2}} = -\frac{1}{32}$$

$$T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

Also remember differentiating and integrating series:

Ex Write $\int_0^t x \ln(1+x^2)\, dx$ as a power series.

2 steps:

First find a series for $x \ln(1+x^2)$:

$$x \ln(1+x^2) = x \ln(1-(-x^2)) = x \sum_{n=1}^{\infty} -\left(\frac{(-x^2)^n}{n}\right)$$

$$= \sum_{n=1}^{\infty} (-x) \frac{(-1)^n x^{2n}}{n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{n}$$

$$\int_0^t x \ln(1+x^2)\, dx = \int_0^t \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{n}\, dx$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+2}}{n(2n+2)} \Bigg|_{x=0}^{x=t}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{t^{2n+2}}{2n+2}$$