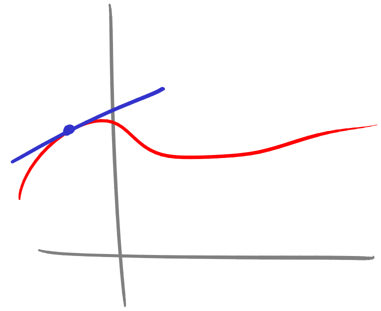
Lecture 2

2 Sep 2014

Last time: parameterized curves $\quad x(t)$, $y(t)$



Slope of tangent line:

$$\text{slope} = \text{"}\frac{dy}{dx}\text{"} = \frac{dy/dt}{dx/dt} \quad \text{as long as this isn't } \frac{0}{0}!$$

$$\left(\text{If it is } \frac{0}{0}, \text{ say at } t=a, \text{ then try } \text{slope}(t=a) = \lim_{t\to a} \frac{dy/dt}{dx/dt}\right)$$

Remark Suppose we want to know the concavity of a parameterized curve — then want to compute "$\frac{d^2y}{dx^2}$". This is not given by $\frac{d^2y/dt^2}{d^2x/dt^2}$ ← WRONG

Rather, we want $\frac{d}{dx}(\text{slope})$

or equivalently $\quad \dfrac{\frac{d}{dt}(\text{slope})}{\frac{dx}{dt}} = \text{"}\frac{d^2y}{dx^2}\text{"}$

[why? rough idea:
we want $\frac{d}{dx}(\text{slope})$
so first compute $\frac{d}{dt}(\text{slope})$
then take $\frac{\cancel{dt}}{dx}\cdot\frac{d}{\cancel{dt}}(\text{slope})$]

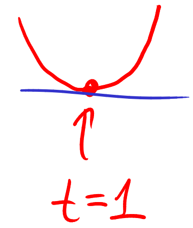
Ex Say have param. curve $\quad x = t^2$, $\quad y = t^3 - 3t$

Is this curve concave up or concave down at $t=1$?

$$\text{slope} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t} = 0 \text{ at } t=1 \quad \text{(horiz tangent)}$$

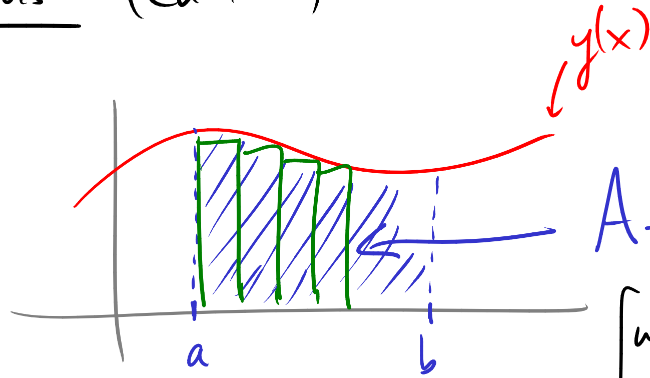
$$\text{"}\frac{d^2y}{dx^2}\text{"} = \frac{d/dt\,(\text{slope})}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t} = \cdots = \frac{3(1+t^2)}{4t^3}$$

So at $t=1$, "$\frac{d^2y}{dx^2}$" $= \frac{6}{4} = \frac{3}{2} > 0$ $\Longrightarrow$ slope is increasing as we move to the right



Areas under curves (Ch 10.2)

Recall:

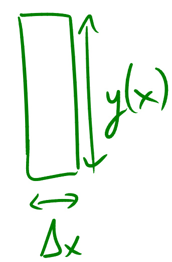


$$A = \int_a^b y(x)\,dx$$

[why? Approximate A by a sum over small rectangles:

each has area

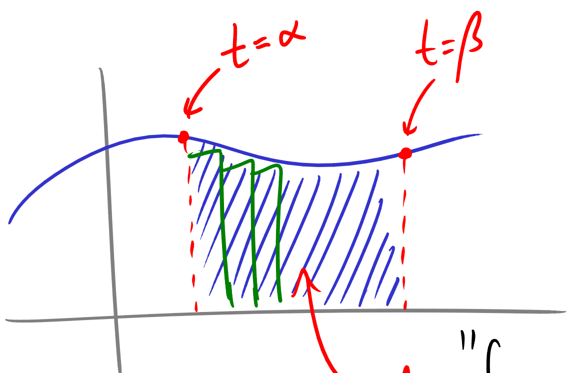
$y(x)\cdot\Delta x$



So $A \approx$ sum of $y(x)\cdot\Delta x$ over all rect.

as $\Delta x \to 0$, this becomes $A = \int y\,dx$]

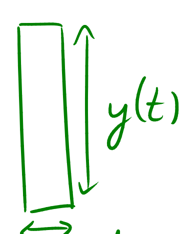
Now say have parameterized curve:



$x = x(t)$
$y = y(t)$

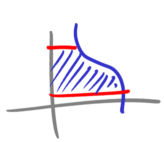
$$A = \text{"}\int y\,dx\text{"} = \int_\alpha^\beta y(t)\,\frac{dx}{dt}\,dt$$

[Why? Essentially same reason:



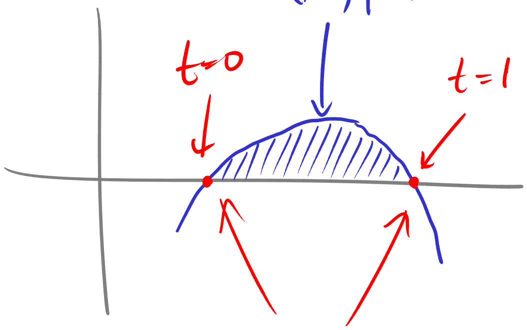
$\Delta x \approx \frac{dx}{dt}\Delta t$]

[Rk for area of region between the curve and the y-axis, would use instead $\int x\,dy$ $\text{"} = \text{"}$ $\int x \frac{dy}{dt}\,dt$]



Ex Find the area of the region between the x-axis and the parameterized curve

$$x = 1 + e^t$$
$$y = t - t^2$$



pts where $y=0$, i.e. $t - t^2 = 0$

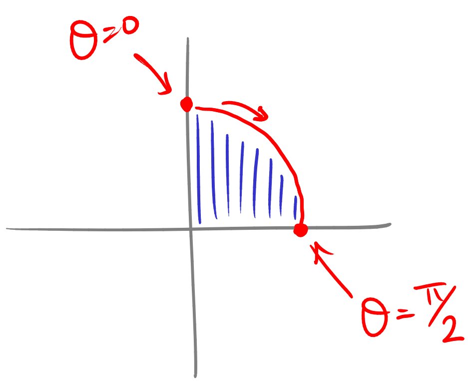
$t(t-1) = 0 \quad t=0,\ t=1$

at $t=0$, $(x,y) = (2,0)$ at $t=1$, $(x,y) = (1+e, 0)$

$$A = \int_0^1 y \cdot \frac{dx}{dt}\, dt = \int_0^1 (t - t^2)\, e^t\, dt = \dots = \underline{3-e}$$

use integration by parts

Ex Find the area of a unit $\frac{1}{4}$-circle, using the description of the circle as a parameterized curve.



$$x = \sin\theta \qquad 0 \le \theta \le \frac{\pi}{2}$$
$$y = \cos\theta$$

$$A = \int_0^{\pi/2} y \cdot \frac{dx}{d\theta}\, d\theta$$

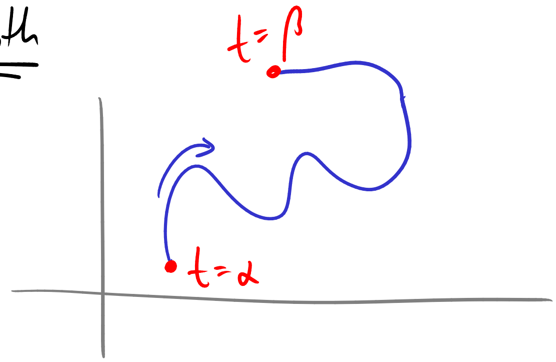
$$= \int_0^{\pi/2} \cos\theta \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\, d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} \Big]_0^{\pi/2}$$

$$= \left(\frac{\pi}{4} + 0\right) - (0+0) = \underline{\underline{\frac{\pi}{4}}}$$

Arc length

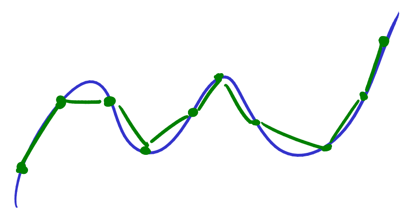


$x(t),\ y(t)$

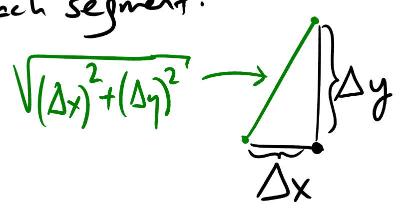
Length: $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

$(\alpha < \beta)$

Why? Approximate path by a polygonal path



Length of each segment:



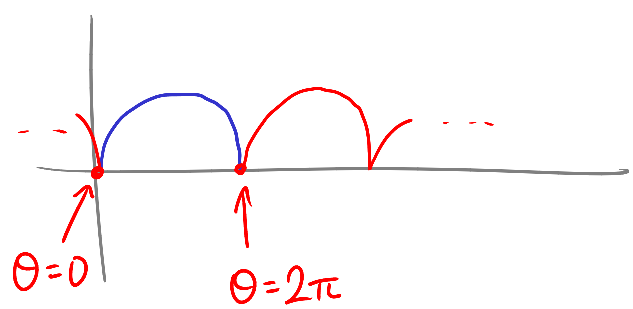
and if we make the segments very small, $\Delta x \approx \frac{dx}{dt} \cdot \Delta t$

$$\Delta y \approx \frac{dy}{dt} \cdot \Delta t$$

length of segment $\approx \sqrt{\left(\frac{dx}{dt}\right)^2 (\Delta t)^2 + \left(\frac{dy}{dt}\right)^2 (\Delta t)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot \Delta t$

sum up all the segments, take $\Delta t \to 0$: $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$

Ex Cycloid: $x = \theta - \sin\theta$
$y = 1 - \cos\theta$



Length of one arch:

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\, d\theta$$

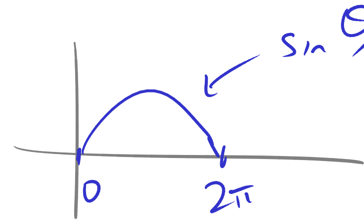
$$= \int_0^{2\pi} \sqrt{1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{=1}}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos\theta}\, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{2\sin^2\frac{\theta}{2}}\, d\theta$$

$[\sqrt{A^2} = |A|]$

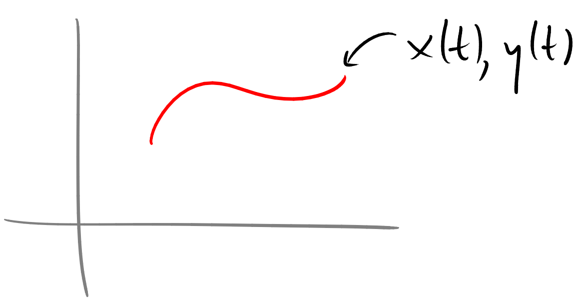


$$= 2 \int_0^{2\pi} \left|\sin\frac{\theta}{2}\right| d\theta$$

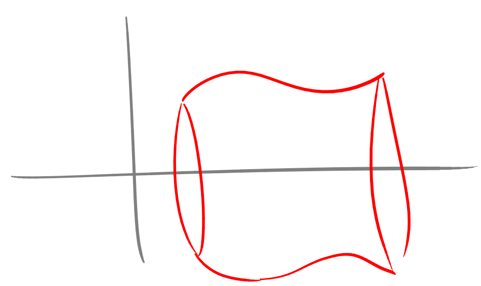
$$= 2 \int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta \qquad \left[\text{since } \sin\frac{\theta}{2} > 0 \text{ for } 0 \le \theta \le 2\pi\right]$$

$$= 2\left(-2\cos\frac{\theta}{2}\Big]_0^{2\pi}\right) = 2(2+2) = \underline{\underline{8}}$$

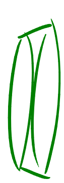
Surface area for surfaces of revolution



What is the surface area of the surface obtained by rotating this curve around the x-axis?



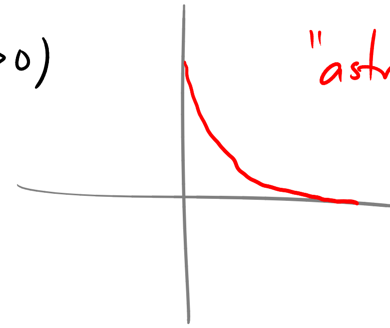
$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

[Why? Cut into cylindrical "ribbons"  with surface area $= \underbrace{2\pi y}_{\text{circumference}} \cdot \underbrace{\sqrt{(\Delta x)^2 + (\Delta y)^2}}_{\text{width}}$]

<u>Ex</u> Take the curve $\quad x = a \cdot \cos^3\theta \qquad (a>0)$

$\qquad\qquad\qquad\qquad\quad y = a \cdot \sin^3\theta$

$\qquad\qquad\qquad\qquad\qquad 0 \le \theta \le \pi/2$



"astroid"

rotate around x-axis:

$$S = \int_0^{\pi/2} 2\pi y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi/2} 2\pi \left(a \sin^3\theta\right) \sqrt{\left(-3a \sin\theta \cos^2\theta\right)^2 + \left(3a \cos\theta \sin^2\theta\right)^2}\, d\theta$$

$$= \int_0^{\pi/2} 2\pi a \sin^3\theta \sqrt{9a^2 \sin^2\theta \cos^4\theta + 9a^2 \cos^2\theta \sin^4\theta}\, d\theta$$

$$= \int_0^{\pi/2} 6\pi a^2 \sin^3\theta \sqrt{\sin^2\theta \cos^2\theta \left(\sin^2\theta + \cos^2\theta\right)}\, d\theta$$

$$= \int_0^{\pi/2} 6\pi a^2 \sin^3\theta \sin\theta \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} 6\pi a^2 \sin^4\theta \cos\theta \, d\theta$$

$$= 6\pi a^2 \left.\frac{\sin^5\theta}{5}\right]_0^{\pi/2} = \frac{6\pi a^2}{5}(1-0) = \underline{\underline{\frac{6\pi a^2}{5}}}$$