Lecture 17

28 Oct 2014

Midterm exam 2: Tuesday Nov 4, ~~11:00~~ 11:05 – 12:15

~~15~~ 14 questions (format the same as midterm 1)

Covers material through Lecture 17 (today)

HW10 due Thursday Nov 6 3am

HW11 due Tuesday Nov 11 3am

⋮

Last time: tangent planes to graphs $z = f(x,y)$

at $(x_0, y_0, f(x_0,y_0))$: $\quad z - z_0 = (x - x_0) f_x(x_0,y_0) + (y - y_0) f_y(x_0,y_0)$

where $f(x_0,y_0) = z_0$

linear approximation: $\quad \Delta z \approx \Delta x \cdot f_x(x_0,y_0) + \Delta y \cdot f_y(x_0,y_0)$

Chain rule for multiple variables (Ch 14.5)

Recall chain rule for functions of one variable:

Say

- y is a function of x $\qquad y = f(x)$
- x is a function of t $\qquad x = g(t)$
- Then y is a function of t: $\qquad y = f(g(t))$

(y depends on t implicitly, through x)

Chain rule: $\displaystyle \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = f'(g(t)) \cdot g'(t)$

With more than one variable, the way variables depend on each other can be more complicated.

Let's start with:

- $z = f(x,y)$
- x, y are both functions of t: $x = g(t)$, $y = h(t)$
- Thus z is a function of t through x and y,

$$z = f(g(t), h(t))$$

Q: what is $\frac{dz}{dt}$?

A: given by multivariable chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Ex Say $z = z(x,y) = x^2 + y^2 + xy$ $\quad$ $x = x(t) = \sin t$, $y = y(t) = e^t$

z is implicitly a function of t.
Let's compute $\frac{dz}{dt}$.

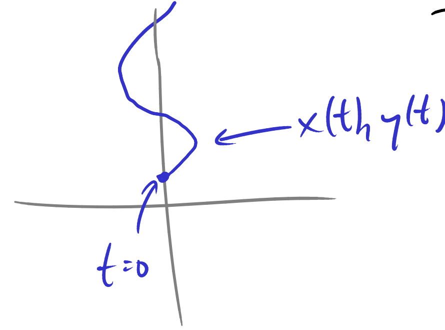
$$\frac{\partial z}{\partial x} = 2x + y \qquad \frac{\partial z}{\partial y} = 2y + x \qquad \frac{dx}{dt} = \cos t \qquad \frac{dy}{dt} = e^t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2x+y)\cos t + (2y+x)e^t$$

$$= (2\sin t + e^t)\cos t + (2e^t + \sin t)e^t$$

$$= \underline{2\sin t \cos t + e^t \cos t + 2e^{2t} + e^t \sin t}$$

What is the meaning of this? Have $z(x,y) = x^2+y^2+xy$



and we study what happens to the function z along the parameterized curve

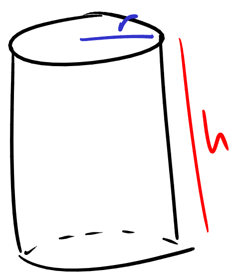
$$x = x(t) = \sin t$$
$$y = y(t) = e^t$$

We could also have substituted directly: $z = \sin^2 t + e^{2t} + e^t \sin t$

$$\frac{dz}{dt} = 2\sin t\cos t + 2e^{2t} + e^t\cos t + e^t \sin t$$

✓ matches what we got from the multi-variable chain rule.

Ex



Volume of cylinder $V = \pi r^2 h$

Suppose r is increasing at the rate 3 cm/s
h is decreasing at the rate 1 cm/s

What is $\frac{dV}{dt}$ when $r = 10$ cm, $h = 20$ cm?

Chain rule: $$\frac{dV}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

$$= (2\pi r h)\cdot\frac{dr}{dt} + (\pi r^2)\cdot\frac{dh}{dt}$$

$$= 2\pi(10\text{ cm})(20\text{ cm})\cdot(3\text{ cm/s}) + \pi(10\text{ cm})^2(-1\text{ cm/s})$$

$$= 1200\pi\text{ cm}^3/\text{s} - 100\pi\text{ cm}^3/\text{s} = \underline{1100\pi\text{ cm}^3/\text{s}}$$

Why is the multivariable chain rule true?

To prove it, use linear approximation: $z = z(x,y)$ $\quad x = x(t)$, $y = y(t)$

we vary t by an amount Δt, want to determine the corresp Δz

$$\Delta z \approx \frac{\partial z}{\partial x} \cdot \Delta x + \frac{\partial z}{\partial y} \cdot \Delta y$$

$$\frac{\Delta z}{\Delta t} \approx \frac{\partial z}{\partial x} \cdot \frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y} \cdot \frac{\Delta y}{\Delta t}$$

In the limit $\Delta t \to 0$ this becomes

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$ as we wanted.

Warning: the mnemonic in 1 variable $\frac{dy}{dt} = \frac{dy}{\cancel{dx}} \cdot \frac{\cancel{dx}}{dt}$

doesn't extend well to 2 variables:

$$\frac{dz}{dt} = \frac{\partial z}{\cancel{\partial x}} \cdot \frac{\cancel{dx}}{dt} + \frac{\partial z}{\cancel{\partial y}} \cdot \frac{\cancel{dy}}{dt} = 2 \cdot \frac{dz}{dt}$$??? wrong!

A mnemonic that does work: take the total differential

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

and divide by dt on both sides.

Another situation: $z = f(x,y)$ $\quad x = g(s,t)$, $y = h(s,t)$

Then $z = f(g(s,t), h(s,t))$ is indirectly a function of s and t.

So, can ask for $\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}$.

e.g. for $\frac{\partial z}{\partial s}$, do just as we did above, treating t as a constant.

So $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

[holding t fixed, holding y fixed, holdy t fixed, x fixed, t fixed]

similarly $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Ex Say $z(x,y) = x^2y^3$ $\quad x(s,t) = s\cos t$, $y(s,t) = s\sin t$

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at $(s,t) = (1,\pi/4)$.

At $(s,t) = (1,\pi/4)$ have $x = 1/\sqrt{2}$ $\quad y = 1/\sqrt{2}$ $\quad z = \frac{1}{4\sqrt{2}}$

$\frac{\partial z}{\partial x} = 2xy^3 = \frac{1}{2}$ $\quad \frac{\partial z}{\partial y} = 3x^2y^2 = \frac{3}{4}$ $\quad \frac{\partial x}{\partial s} = \cos t = \frac{1}{\sqrt{2}}$ $\quad \frac{\partial y}{\partial s} = \sin t = \frac{1}{\sqrt{2}}$ $\quad \frac{\partial x}{\partial t} = -s\sin t = -\frac{1}{\sqrt{2}}$ $\quad \frac{\partial y}{\partial t} = s\cos t = \frac{1}{\sqrt{2}}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{3}{4}\cdot\frac{1}{\sqrt{2}} = \underline{\underline{\frac{5}{4\sqrt{2}}}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{1}{2}\left(-\frac{1}{\sqrt{2}}\right) + \frac{3}{4}\cdot\frac{1}{\sqrt{2}} = \underline{\underline{\frac{1}{4\sqrt{2}}}}$$

General case: $\qquad z = f(x_1, x_2, \dots, x_n)$

Each x_i depends on $(t_1, \dots, t_m)$ $\qquad x_i = g_i(t_1, \dots, t_m)$

Then z depends indirectly on $(t_1, \dots, t_m)$ <u>through</u> $(x_1, \dots, x_n)$

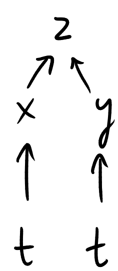
And
$$\frac{\partial z}{\partial t_1} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_1}$$
$$\frac{\partial z}{\partial t_2} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_2}$$
$$\vdots$$
$$\frac{\partial z}{\partial t_m} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

(This would look prettier in the language of matrices!)

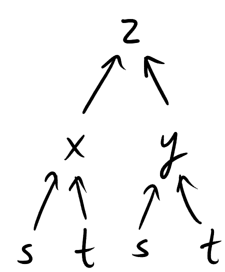
Can organize the dependences in terms of a tree diagram:

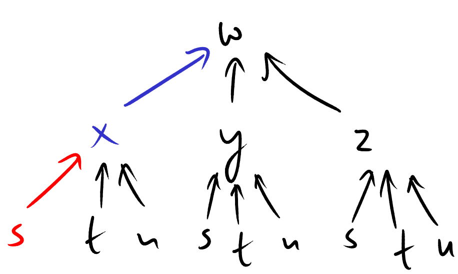
e.g. $y = y(x)$, $x = x(t)$

$z = z(x,y)$, $x = x(t)$, $y = y(t)$



$z = z(x,y)$, $x = x(s,t)$, $y = y(s,t)$





$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Multivariable chain rule gives another way of understanding implicit differentiation.

Recall: If we have a curve in the plane defined by

$$F(x,y) = 0 \qquad (*)$$

it determines y as an implicit function of x. $\qquad y = y(x)$

Q: what is $\frac{dy}{dx}$?

$$F(x, y(x)) = 0$$

$$\frac{d}{dx} F(x, y(x)) = 0$$

$$\frac{\partial F}{\partial x} \cdot \underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$$