Lecture 18

30 Oct 2014

Exam 2 material: lectures 10–17
(Sep 30 – Oct 28)

Last time: Chain rule for multivariable functions

e.g. $F(x,y,z)$ $\quad$ $x = x(s,t)$
$y = y(s,t)$
$z = z(s,t)$

(holding t fixed)

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial s}$$

Implicit diff. via the chain rule:

Suppose we have $F(x,y,z) = 0$ determining $z = z(x,y)$

and want to determine $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

$$F(x,y,z(x,y)) = 0$$

Now define a new function of two variables $G(x,y) = F(x,y,z(x,y)) = 0$

Let's compute $\frac{\partial}{\partial x} G(x,y)$ by the chain rule.

We get:

$$\underbrace{\frac{\partial}{\partial x} G(x,y)}_{\substack{\parallel \\ 0 \text{ because } G(x,y)=0}} = \frac{\partial F}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial x}}_{1} + \frac{\partial F}{\partial y}\cdot\underbrace{\frac{\partial y}{\partial x}}_{0} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}$$

so we get

$$0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}$$

i.e.

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}$$

Directional Derivatives (Ch 14.6)

$f(x,y)$

We've talked a lot about f_x, f_y

f_x = "rate of change in the x-dir" $\bullet\!\longrightarrow$

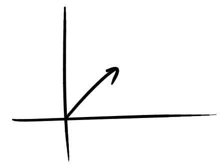
f_y = "rate of change in the y-dir" $\uparrow$

How about rate of change in some <u>other</u> <u>direction</u>?

To specify a direction, pick a <u>unit</u> vector in that direction $\vec{u}$

e.g. x-dir $\vec{u} = \langle 1,0\rangle$

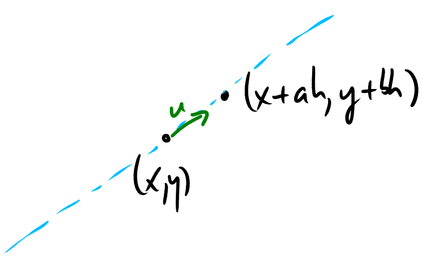
y-dir $\vec{u} = \langle 0,1\rangle$



$\vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

Say $\vec{u} = \langle a,b\rangle$

Then define $$D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+a\cdot h, y+b\cdot h) - f(x,y)}{h}$$



"directional derivative of f in the direction $\vec{u}$"

e.g. if $\vec{u} = \langle 1,0\rangle$ $D_{\vec{u}} f = f_x$

if $\vec{u} = \langle 0,1\rangle$ $D_{\vec{u}} f = f_y$.

<u>Fact</u> (If f is differentiable), if $\vec{u} = \langle a,b\rangle$

$$D_{\vec{u}} f = a\cdot f_x + b\cdot f_y$$

Why? Say $g(h) = f(x+ah, y+bh)$

Then $D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h} = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = g'(0)$

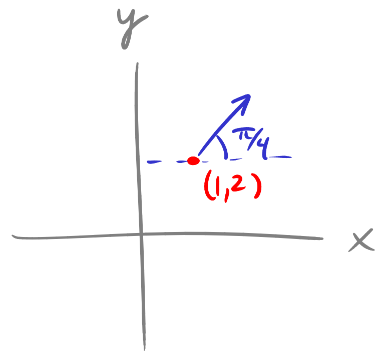
By chain rule $\frac{d}{dh} g(h) = \frac{d}{dh} f(x+ah, y+bh) = f_x(x+ah, y+bh)\cdot a + f_y(x+ah, y+bh)\cdot b$

$$\left[\quad \text{So} \quad g'(0) = f_x(x,y)\cdot a + f_y(x,y)\cdot b \quad \right]$$

<u>Ex</u> If $f(x,y) = x^3 - xy + 4y^2$

and $\vec{u}$ is unit vector with angle $\pi/4$ to x-axis

what is $D_{\vec{u}} f(1,2)$?



$$\vec{u} = \left\langle \cos \tfrac{\pi}{4},\ \sin \tfrac{\pi}{4} \right\rangle = \left\langle \underset{a}{\frac{1}{\sqrt{2}}},\ \underset{b}{\frac{1}{\sqrt{2}}} \right\rangle$$

$$f_x = 3x^2 - y \qquad f_x(1,2) = 1$$

$$f_y = -x + 8y \qquad f_y(1,2) = 15$$

$$D_{\vec{u}} f(1,2) = a\cdot f_x(1,2) + b\cdot f_y(1,2)$$

$$= \frac{1}{\sqrt{2}}\cdot 1 + \frac{1}{\sqrt{2}}\cdot 15 = \frac{16}{\sqrt{2}} = \frac{16\sqrt{2}}{2} = 8\sqrt{2}$$

<u>Gradient vector</u> Note $D_{\vec{u}} f = a\cdot f_x + b\cdot f_y$

$$= \langle a, b\rangle \cdot \langle f_x, f_y\rangle$$

$$= \vec{u} \cdot \langle f_x, f_y\rangle$$

So, define

$$\vec{\nabla} f = \langle f_x, f_y\rangle \qquad \text{"gradient vector of } f\text{"}$$

Then $D_{\vec{u}} f = \vec{u} \cdot \vec{\nabla} f$

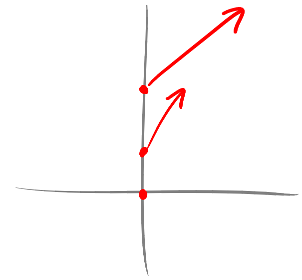
<u>Ex</u> If $f(x,y) = e^x y^2$

then $\vec{\nabla} f = \langle f_x, f_y\rangle = \langle e^x y^2,\ 2e^x y\rangle$

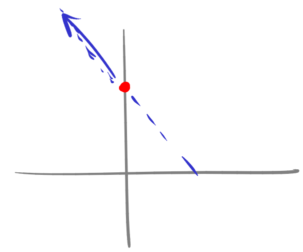
$(x,y)=(0,2)$: $\vec{\nabla} f(0,2) = \langle 4,4 \rangle$

$\vec{\nabla} f(0,1) = \langle 1,2 \rangle$

$\vec{\nabla} f(0,0) = \langle 0,0 \rangle$



Ex What is the directional derivative of $f(x,y) = e^x y^2$ at $(0,2)$ in the direction of the vector $\langle -3,4 \rangle$?



$$\vec{u} = \frac{\langle -3,4 \rangle}{\|\langle -3,4 \rangle\|} = \frac{\langle -3,4 \rangle}{5} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$$

$$D_{\vec{u}} f = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle \cdot \langle 4,4 \rangle = -\frac{12}{5} + \frac{16}{5} = \underline{\underline{\frac{4}{5}}}$$

We did all this in 2 dimensions $f(x,y)$
but all works just the same in 3d $f(x,y,z)$

e.g. $\vec{u} = \langle a,b,c \rangle$

$$D_{\vec{u}} f(x,y,z) = \lim_{h \to 0} \frac{f(x+ah, y+bh, z+ch) - f(x,y,z)}{h}$$

$$= \vec{u} \cdot \vec{\nabla} f$$

$$\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$$

Ex If $f(x,y,z) = \frac{y^2 z}{x}$ find $\vec{\nabla} f$ at $(-2,3,5) = (x,y,z)$ and the dir. deriv. of f in direction $\langle 1,2,0 \rangle$

$$\vec{\nabla} f = \left\langle -\frac{y^2 z}{x^2}, \frac{2yz}{x}, \frac{y^2}{x} \right\rangle \quad \text{so } \vec{\nabla} f(-2,3,5) = \left\langle -\frac{45}{4}, -15, -\frac{9}{2} \right\rangle$$

$$\vec{u} = \frac{\langle 1,2,0 \rangle}{\|\langle 1,2,0 \rangle\|} = \frac{\langle 1,2,0 \rangle}{\sqrt{5}}$$

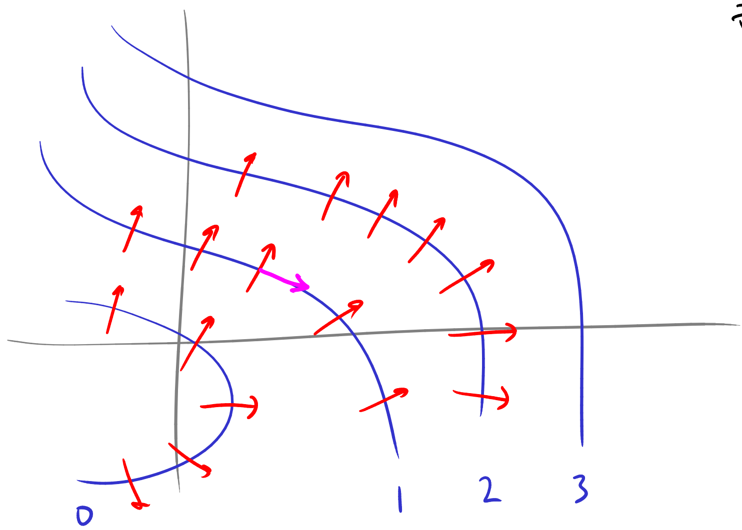
at $(-2,3,5)$ $\quad D_{\vec{u}} f = \vec{u} \cdot \vec{\nabla} f = \frac{\langle 1,2,0 \rangle}{\sqrt{5}} \cdot \left\langle -\frac{45}{4}, -15, -\frac{9}{2} \right\rangle$

$$= \frac{1}{\sqrt{5}} \left(-\frac{45}{4} - 30 + 0 \right)$$

$$= \frac{1}{\sqrt{5}} \left(-\frac{165}{4} \right) = -\frac{165}{4\sqrt{5}}$$

How to interpret / think about $\vec{\nabla} f$?

Fact



$\vec{\nabla} f$ is always $\perp$ to contour lines.

Why? If we walk along a path tangent to contour line then f is constant. So, if $\vec{u}$ is tangent to contour line then $D_{\vec{u}} f = 0$.

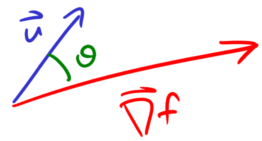
But $D_{\vec{u}} f = \vec{u} \cdot \vec{\nabla} f = 0$

so $\vec{u} \perp \vec{\nabla} f$.

Fact The unit vector $\vec{u}$ for which $D_{\vec{u}} f$ is maximum is the vector $\vec{u} = \frac{\vec{\nabla} f}{\|\vec{\nabla} f\|}$. $\quad$ (if $\vec{\nabla} f \neq \langle 0,0 \rangle$)

("$\vec{\nabla} f$ tells you which way to walk to increase f the fastest")

Why? $D_{\vec{u}} f = \vec{u} \cdot \vec{\nabla} f$



$$= \|\vec{u}\| \cdot \|\vec{\nabla} f\| \cdot \cos\theta$$

$$= \|\vec{\nabla} f\| \cdot \cos\theta$$

maximized when $\cos\theta = 1$, i.e. $\theta = 0$

i.e. $\vec{u}$ is parallel to $\vec{\nabla} f$.

i.e. $\vec{u} = \dfrac{\vec{\nabla} f}{\|\vec{\nabla} f\|}$

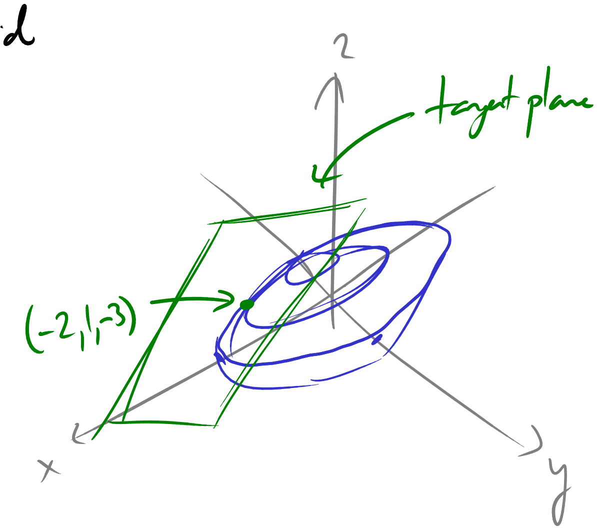
Similarly $\dfrac{-\vec{\nabla} f}{\|\vec{\nabla} f\|} = \vec{u}$ gives the direction of fastest decrease.

Fact If $f = f(x,y,z)$ have level surfaces in 3d (rather than level curves in 2d) and again, $\vec{\nabla} f$ is $\perp$ to the level surfaces.

Ex Find the tangent plane to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

at $(x,y,z) = (-2,1,-3)$.



How to find a normal vector to this plane?

The ellipsoid is a level surface: $F(x,y,z) = 3$

where $F(x,y,z) = \dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9}$

Thus $\vec{\nabla} F$ is $\perp$ to the ellipsoid $\Longrightarrow$ also $\perp$ to tangent plane

$$\vec{\nabla} F = \left\langle \frac{x}{2}, 2y, \frac{2}{9}z \right\rangle$$

$$\vec{\nabla} F(-2,1,-3) = \left\langle -1, 2, -\frac{2}{3} \right\rangle$$

So, we want plane through $(-2,1,-3)$ w/ normal vector $\left\langle -1, 2, -\frac{2}{3} \right\rangle$

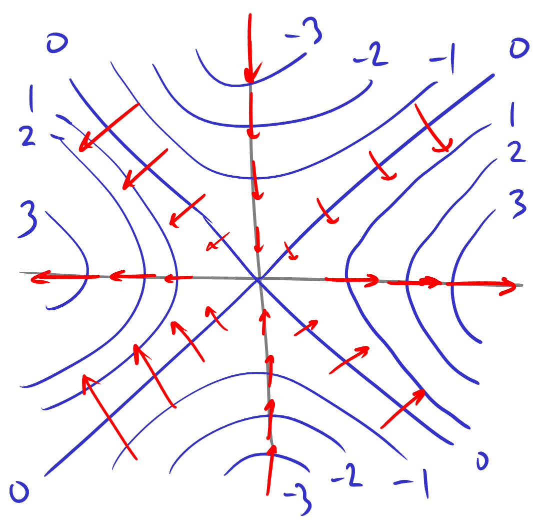
$$-1(x+2) + 2(y-1) - \frac{2}{3}(z+3) = 0$$

$$-x-2+2y-2-\frac{2}{3}z-2=0$$

$$\underline{-x+2y-\frac{2}{3}z = 6}$$

Rk $f(x,y) = x^2 - y^2$

$\vec{\nabla} f = \langle 2x, -2y \rangle$



Illustrates the phenomenon: if contour lines cross, $\underline{\vec{\nabla} f = 0 \text{ there}}$.