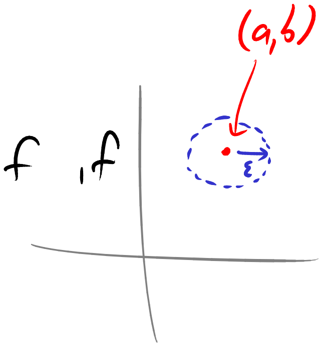
Lecture 19

16 Nov 2014

Final exam Sat Dec 13 9–12

HW11 due this Tuesday (Nov 11)

Maxima and Minima for functions of 2 variables (Ch 14.7)

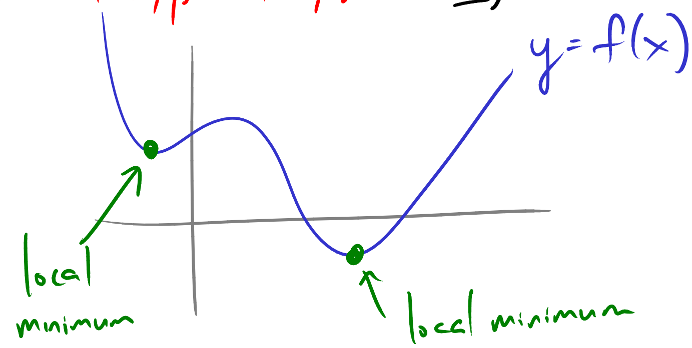


Def If we have a function $f(x,y)$ then we say (a,b) is a local minimum (or max) for f if there is some ε such that, for all points (x,y) in a disc of radius ε around (a,b),

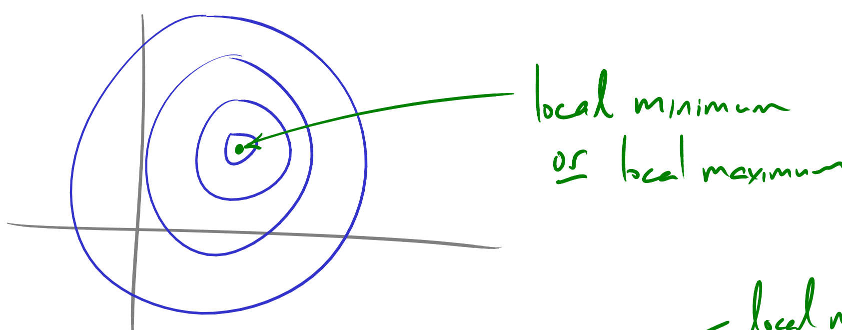
$$f(x,y) \geq f(a,b).$$

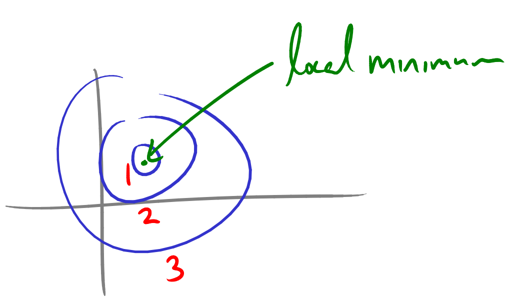
(or $f(x,y) \leq f(a,b)$ for max)

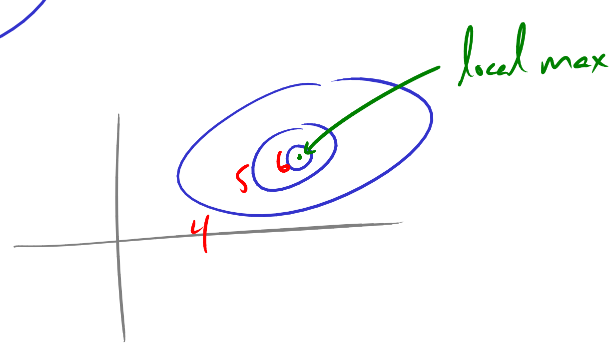
Recall the picture in 1 variable:



Contour map:







Def If $\partial_x f(a,b) = 0$ and $\partial_y f(a,b) = 0$ then we call (a,b) a <u>critical point</u> of f.

Fact (If f is differentiable), <u>if</u> (a,b) is a local min/max for f, <u>then</u> (a,b) is a critical point for f.

Ex Find the local minima and maxima of

$$f(x,y) = x^2 + y^2 - 2x - 6y + 14.$$

Find critical points:

$$f_x = 2x - 2 = 0$$
$$f_y = 2y - 6 = 0$$

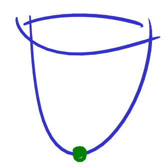
Solve for (x,y): $x = 1,\ y = 3$

$\Rightarrow$ only <u>one</u> critical point, at $(1,3)$.

$$f(1,3) = 1 + 9 - 2 - 18 + 14 = \underline{4}$$

Is it a local max, local min, or neither?

<u>Local min</u> – e.g. from visualizing the graph



or rewriting $f(x,y) = (x-1)^2 + (y-3)^2 + 4$

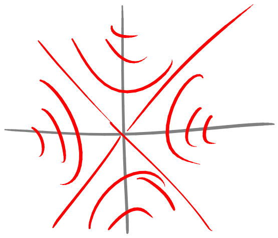
Ex Find local min/local max for

$$f(x,y) = x^2 - y^2$$

Critical pts:

$$f_x = 2x = 0$$
$$f_y = -2y = 0$$

solve for (x,y): $(x,y) = (0,0)$



$(0,0)$ is a saddle point (cf last lecture)
not a local max/min.
So $f(x,y)$ has no local max/min.

Second derivative test (If f_{xx}, f_{yy}, f_{xy} are continuous near (a,b))

If (a,b) is a critical point of f:

let $D = f_{xx} f_{yy} - f_{xy}^2$ $\quad \left(= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} \right)$

- If $D > 0$, $\begin{cases} \text{if } f_{xx} > 0 \text{ then } (a,b) \text{ is local min} \\ \text{if } f_{xx} < 0 \text{ then } (a,b) \text{ is local max} \end{cases}$

- If $D < 0$, (a,b) is saddle point
- If $D = 0$, test is inconclusive

Ex If $f(x,y) = x^2 + y^2 - 2x - 6y + 14$

$f_x = 2x - 2 \qquad f_y = 2y - 6 \qquad (x,y) = (1,3)$ crit. pt.

$f_{xx} = 2 \qquad f_{xy} = 0 \qquad f_{yy} = 2$

$D = f_{xx} f_{yy} - f_{xy}^2 = 2 \cdot 2 - 0 = 4 > 0$

and $f_{xx} = 2 > 0$ $\qquad$ So, $(1,3)$ is local minimum

Ex If $f(x,y) = x^2 + 3y^2 - 10xy$

$f_x = 2x - 10y \quad f_y = 6y - 10x$

$f_{xx} = 2$ $\quad$ $f_{xy} = -10$ $\quad$ $f_{yy} = 6$

critical points: $\quad 2x - 10y = 0$

$\quad 6y - 10x = 0$

solve for (x,y): $\quad x = 0, \ y = 0$

$\Longrightarrow$ only critical point is $(0,0)$.

2^{nd} deriv test: $\quad D = \begin{vmatrix} 2 & -10 \\ -10 & 6 \end{vmatrix} = 12 - 100 = \underline{-88 < 0}$

$\Longrightarrow$ $(0,0)$ is $\underline{\text{saddle point}}$.

Why does 2^{nd}-deriv test work?

e.g., suppose $D > 0$, $f_{xx} > 0$. $\quad \vec{u} = \langle h, k \rangle$

$$D_{\vec{u}} f = h \cdot f_x + k \cdot f_y = 0 \quad \text{at critical pt.}$$

$$D_{\vec{u}} D_{\vec{u}} f = h \cdot (h f_{xx} + k \cdot f_{yx}) + k \cdot (h f_{xy} + k \cdot f_{yy})$$

$$= h^2 f_{xx} + hk f_{yx} + hk f_{xy} + k^2 f_{yy}$$

$$= h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}$$

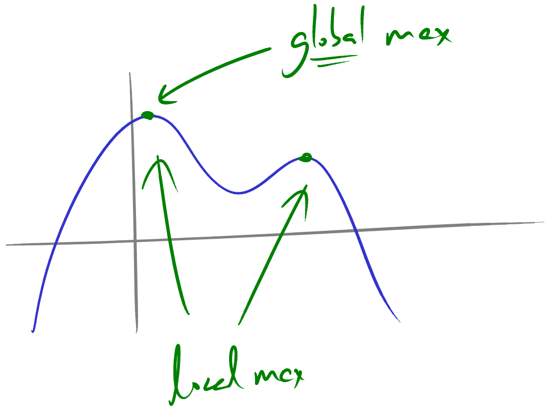
$$= \underbrace{f_{xx}}_{>0} \underbrace{\left(h + \frac{f_{xy}}{f_{xx}} k\right)^2}_{\geq 0 \ [=0 \text{ only if } h=0]} + \underbrace{\frac{k^2}{f_{xx}}}_{\geq 0 \ [=0 \text{ only if } k=0]} \underbrace{\left(f_{xx} f_{yy} - f_{xy}^2\right)}_{>0}$$

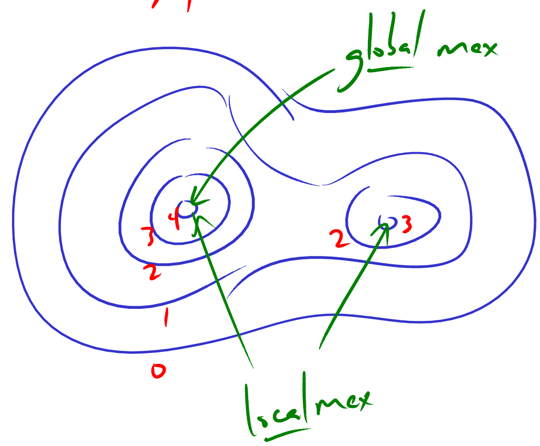
$$> 0$$

$\Longrightarrow$ (a,b) is a $\underline{\text{local minimum}}$!

Often we want *global* max/min, not *local*.

Def If have $f(x,y)$ defined on some domain D in (x,y)-plane we say (a,b) is a *global* maximum for f if, for *any* (x,y) in D, $f(a,b) \geq f(x,y)$.





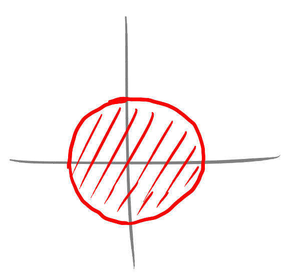
Fact If D is *bounded* (doesn't go off to ∞ in any direction)
closed (contains all its boundary points)

and f is continuous on D

Then f has a global max, and a global min, on D.

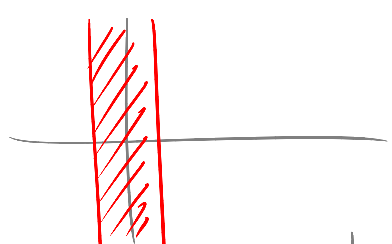
Ex $D = \{x,y : x^2+y^2 \leq 1\}$

is bounded and closed

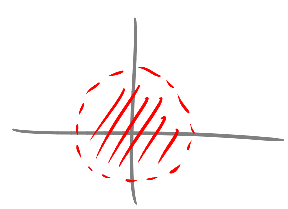


$D = \{x,y : |x| \leq 1\}$

closed, *not* bounded



$D = \{x,y : x^2+y^2 < 1\}$

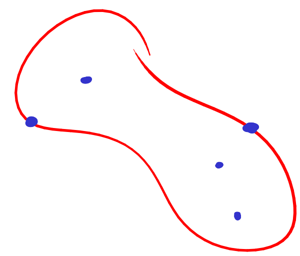


bounded, not closed

[So a function on D doesn't have to have a max,
e.g. $f(x,y) = x^2+y^2$ doesn't —
it does have 1 critical pt, at $(0,0)$
which is global min]

So, suppose D is closed and bounded.

To find the global maximum of f on D:



1. Find all critical points of f on the interior of D, find the values of f at these points.

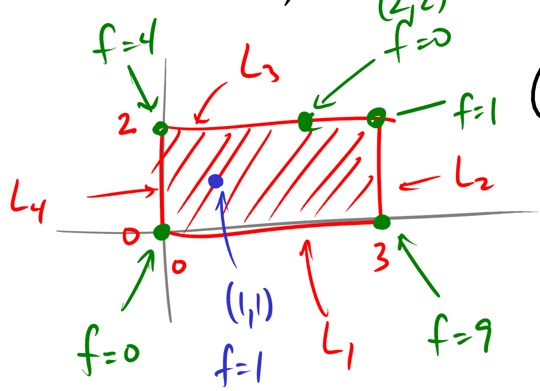
2. Find the maximum value of f on the boundary of D.

3. Take the biggest value of f found in steps 1, 2.

(Similarly for absolute minimum.)

Ex $f(x,y) = x^2 - 2xy + 2y$

Find absolute min, max of $f(x,y)$ on $D = \{0 \le x \le 3,\ 0 \le y \le 2\}$



1. Critical pts: $f_x = 2x - 2y = 0$
$f_y = -2x + 2 = 0$

$\longrightarrow x = 1,\ y = 1$

One crit pt, at $(1,1)$

$f(1,1) = 1$

② On boundary: 4 pieces L_1: $y=0$, $0 \le x \le 3$

$$f(x,0) = x^2$$

min: $f(0,0)=0$

max: $f(3,0)=9$

L_2: $x=3$, $0 \le y \le 2$ $\quad f(3,y) = 9-4y$ $\quad$ min: $f(3,2)=1$

max: $f(3,0)=9$

L_3: $y=2$, $0 \le x \le 3$ $\quad f(x,2) = x^2-4x+4$

need to find min, max of this for $0 \le x \le 3$

— now just a function of one variable x

$f' = 2x-4 \longrightarrow$ crit pt at $x=2$

$f(2,2)=0$

also $f(0,2)=4$

$f(3,2)=1$

L_4: $x=0$, $0 \le y \le 2$ $\quad f(0,y) = 2y$

$f(0,0)=0$ $\qquad f(0,2)=4$

$\Longrightarrow$ global max is 9: $f(3,0)=9$

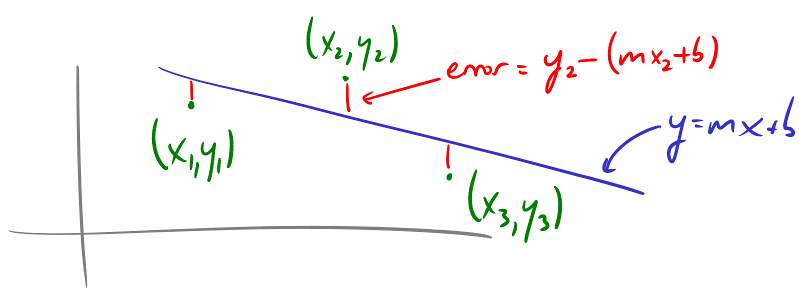
global min is 0: $f(0,0)=0$

$f(2,2)=0$

An application: least-squares fitting

Say we have some data points

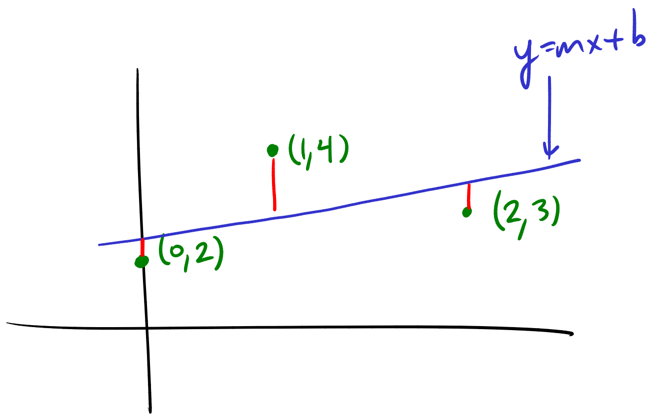
and want to find the "best-fit line."



Consider squared error $E = (y_1 - mx_1 - b)^2 + (y_2 - mx_2 - b)^2 + \cdots + (y_n - mx_n - b)^2$

"Best-fit line": the line which minimizes E

Ex



What is the best-fit line to these 3 data points?

$$E = (2-b)^2 + (4-m-b)^2 + (3-2m-b)^2$$

$$= (4-4b+b^2) + (16+m^2+b^2-8m-8b+2bm) + (9+4m^2+b^2-12m-6b+4bm)$$

$$= 3b^2 + 5m^2 + 6bm - 20m - 18b + 29$$

Critical points:

$$E_b = 6b + 6m - 18 = 0$$

$$E_x = 6b + 10m - 20 = 0$$

$$-4m + 2 = 0 \longrightarrow m = \frac{1}{2}$$

$$6b + 6\left(\tfrac{1}{2}\right) - 18 = 0$$

$$6b - 15 = 0 \longrightarrow b = \frac{5}{2}$$

So best-fit line is $\underline{\underline{y = \frac{1}{2}x + \frac{5}{2}}}$