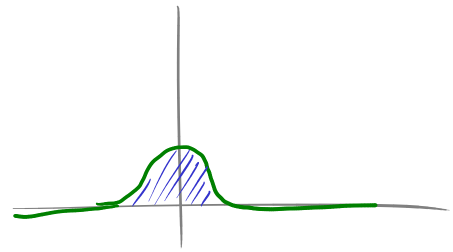
Lecture 22

18 Nov 2014

A puzzle: What is $\int_{-\infty}^{\infty} e^{-x^2}\,dx$?

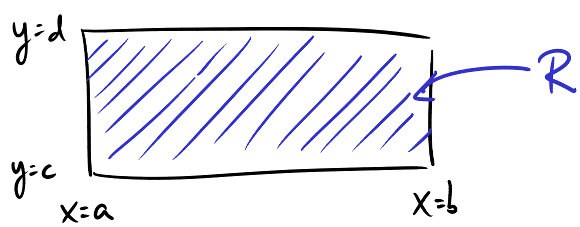


Answer later this lecture/next — an amazing trick using integrals in two variables: let $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$

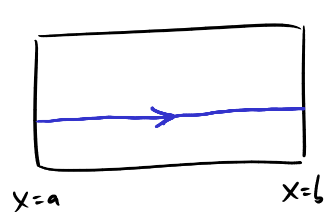
Then $$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$$
$$= \iint_D e^{-x^2-y^2}\,dx\,dy \qquad D = \text{whole } x\text{-}y \text{ plane}$$

And there's a trick for evaluating this!

Last time: double $\int$ over rectangles

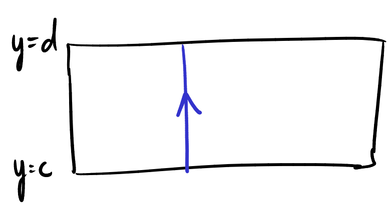


$$\iint_R f(x,y)\,dA = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$$



(slicing by fixed-y slices, then summing up contributions from each slice)

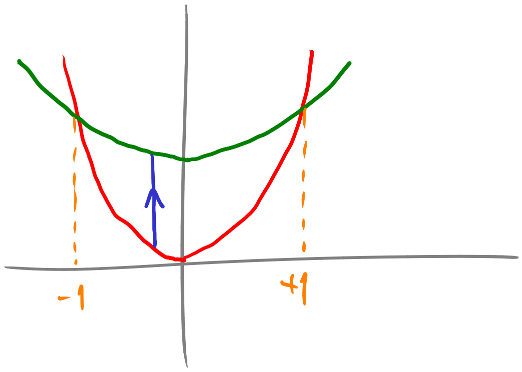
$$= \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx$$



(slicing by fixed-x slices)

How about non-rectangular domains?

Ex Evaluate $\iint_D (x+2y)\,dA$ where D is the domain lying between $y = 2x^2$ and $y = 1+x^2$



If we fix x, y runs from $2x^2$ to $1+x^2$

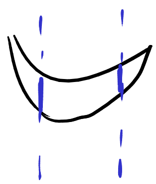
Our integral is $\int_{-1}^{1}\left[\int_{2x^2}^{1+x^2} (x+2y)\,dy\right]dx$

$$= \int_{-1}^{1} xy + y^2 \Big]_{y=2x^2}^{y=1+x^2} dx$$

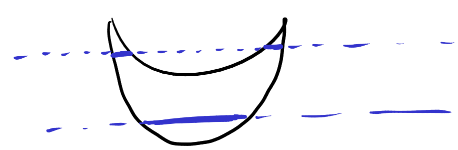
$$= \int_{-1}^{1} \left[x(1+x^2) + (1+x^2)^2\right] - \left[x(2x^2) + (2x^2)^2\right] dx$$

$$= \cdots \frac{32}{15}$$

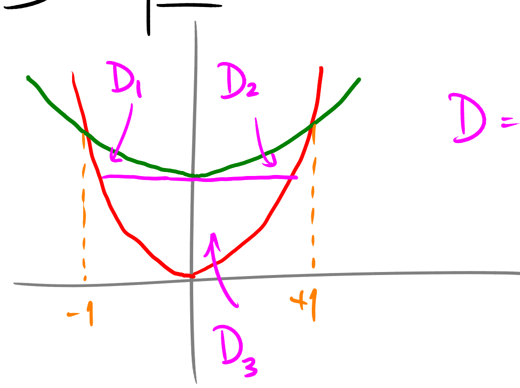
This domain was specially nice for vertical slicing — when we look at fixed x we always get a single interval



But the horizontal slices (y=fixed) are more complicated!



To do this $\int$ by horizontal slices we'd have to break D into pieces



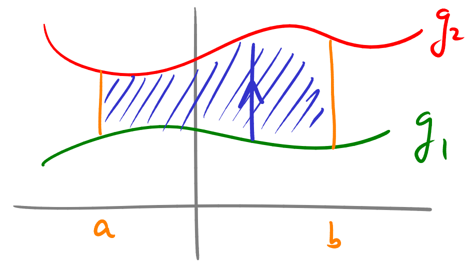
$D = D_1 \cup D_2 \cup D_3$

Whenever we have a domain of the form

$$D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

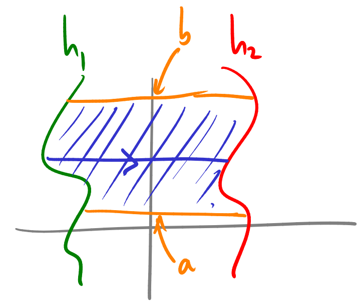
have

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx$$



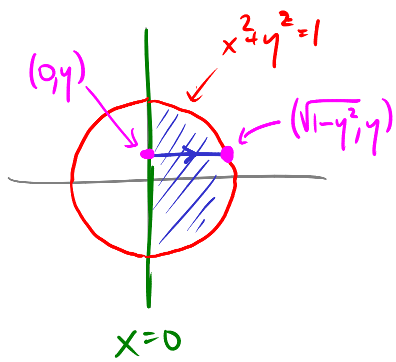
If $D = \{(x,y) : a \le y \le b,\ h_1(y) \le x \le h_2(y)\}$

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy$$



Ex $\iint_D xy^2\,dA$ D = the domain enclosed by the line $x=0$, the circle $x^2+y^2=1$ with $x>0$

$$= \int_{-1}^{1}\left[\int_0^{\sqrt{1-y^2}} xy^2\,dx\right] dy$$



$x^2+y^2=1$

$x=\sqrt{1-y^2}$

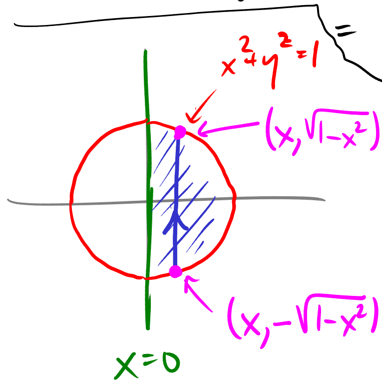
$$= \int_{-1}^{1} \frac{1}{2}x^2y^2 \Big]_{x=0}^{x=\sqrt{1-y^2}} dy$$

$$= \int_{-1}^{1} \frac{1}{2}(1-y^2)y^2 - 0\ dy$$

$$= \int_{-1}^{1} \frac{1}{2}y^2 - \frac{1}{2}y^4\,dy = \frac{1}{6}y^3 - \frac{1}{10}y^5\Big]_{y=-1}^{y=1} = \left(\frac{1}{6}-\frac{1}{10}\right) - \left(-\frac{1}{6}+\frac{1}{10}\right)$$

$$= 2\cdot\left(\frac{1}{6}-\frac{1}{10}\right) = \frac{1}{3}-\frac{1}{5} = \underline{\frac{2}{15}}$$

Or: slice vertically



$$\int_0^1 \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy^2 \, dy \right] dx = \cdots = \underline{\underline{\frac{2}{15}}}$$

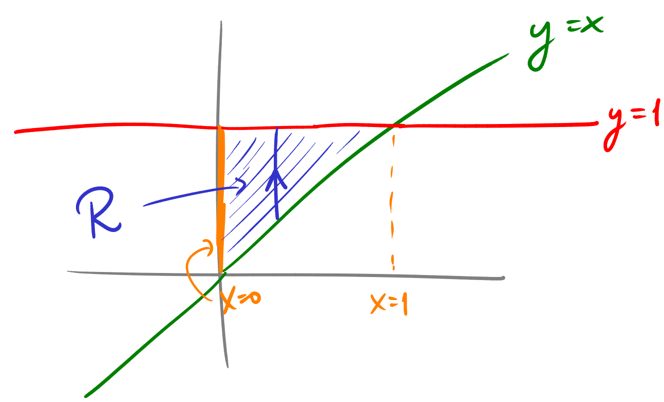
Some painful u-sub here

So the horiz. slices are easier in this case

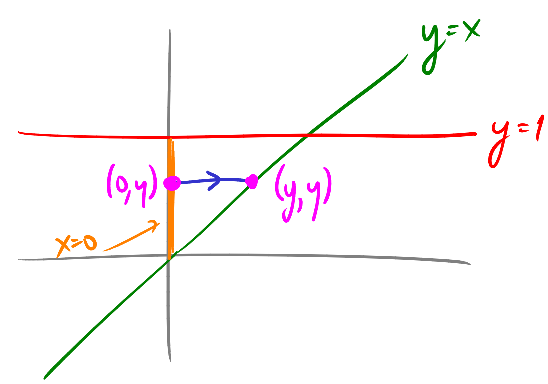
Ex $\int_0^1 \left[\int_x^1 \sin(y^2)\, dy \right] dx$ This looks hopeless at first — can't do the integral over y.

But, we can interpret it as a double integral:

$$\iint_R \sin(y^2) \, dA$$



and now do the same integral by horizontal slices:



$$\int_0^1 \left[\int_0^y \sin(y^2) \, dx \right] dy$$

$$= \int_0^1 x\sin(y^2) \Big]_{x=0}^{x=y} dy$$

$$= \int_0^1 y\sin(y^2) - 0 \;\; dy$$

$u = y^2 \quad du = 2y\,dy \quad dy = \frac{du}{2y}$

$$= \int_0^1 \frac{1}{2} \sin(u) \, du$$

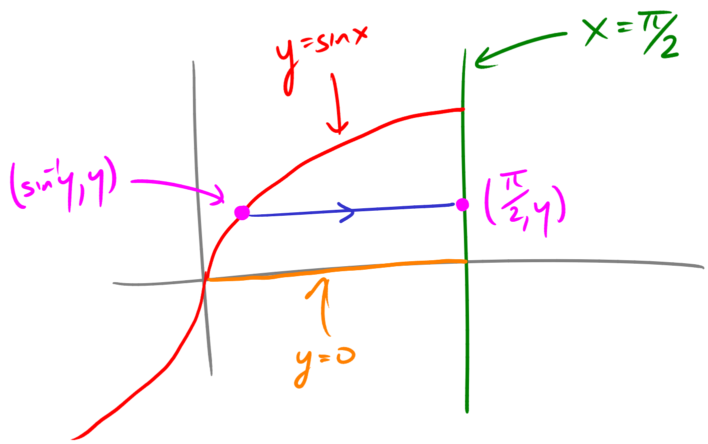
$$= -\frac{1}{2}\cos(u)\Big]_0^1 = -\frac{1}{2}\left(\cos(1)-1\right) = \underline{\frac{1}{2}\left(1-\cos(1)\right)}$$

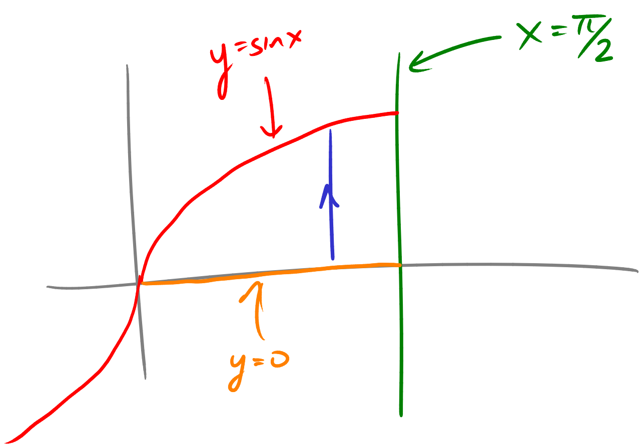
Ex Compute the iterated integral

$$\int_0^1\left[\int_{\sin^{-1}(y)}^{\pi/2}\cos x\sqrt{1+\cos^2 x}\,dx\right]dy$$

Looks hard — see if it's easier when we <u>reverse</u> the order of $\int$.

$x=\sin^{-1}(y)$
means
$y=\sin(x)$
so...





$$\int_0^{\pi/2}\left[\int_0^{\sin x}\cos x\sqrt{1+\cos^2 x}\,dy\right]dx$$

$$= \int_0^{\pi/2}\cos x\sqrt{1+\cos^2 x}\;y\Big]_{y=0}^{y=\sin x}dx$$

$$= \int_0^{\pi/2}\cos x\,\sin x\sqrt{1+\cos^2 x}\,dx$$

$u=\cos x$

$du=-\sin x\,dy$

$$= -\int_1^0\sqrt{1+u^2}\;u\,du$$

$v=1+u^2$

$dv=2u\,du$

$$= \cdots = \underline{\frac{1}{3}\left(2^{3/2}-1\right)}$$

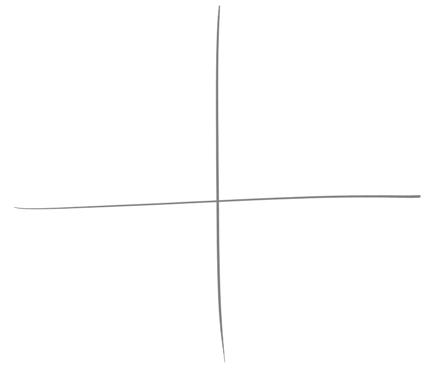
So far mainly review from 408L.

Now, something new:

Let's look again at $$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

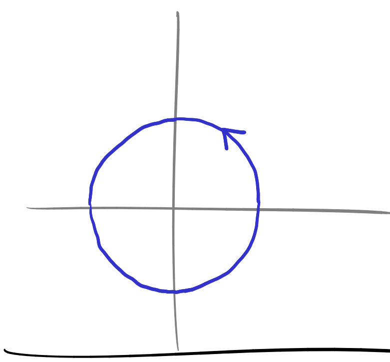
$$I^2 = \iint_D e^{-x^2-y^2}\,dA$$

D = whole x-y plane



We should really write this in polar coordinates (r,θ) $\quad x^2+y^2=r^2$

$$I^2 = \iint_D e^{-r^2}\,dA$$



Now we want to "do the integral over θ first" then the integral over r

$$I^2 = \int_{r=0}^{r=\infty}\int_{\theta=0}^{\theta=2\pi} e^{-r^2}\,\overbrace{r\,d\theta\;\,dr}^{dA}$$

$$= \int_0^\infty \left(e^{-r^2} r\,\theta \Big]_{\theta=0}^{\theta=2\pi} \right) dr$$

$$= \int_0^\infty 2\pi r e^{-r^2}\,dr$$

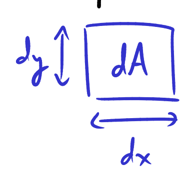
$u = r^2$

$du = 2r\,dr$

$$= \int_0^\infty \pi e^{-u}\,du = -\pi e^{-u}\Big]_{u=0}^{u=\infty}$$

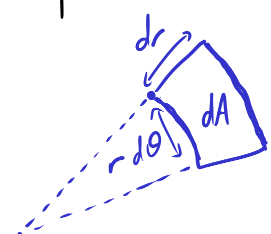
$$= -\pi(0-1)$$

In x-y coords:



$dA = dx\,dy$

In polar coords:



$dA = (dr)(r\,d\theta)$
$= r\,dr\,d\theta$

$= \pi$

So, $\underline{\underline{I = \sqrt{\pi}}}$