Linear Transformation Exercises

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1. Determine whether the following functions are linear transformations. If they are, prove it; if not, provide a counterexample to one of the properties:

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$, with
   $$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ y \end{bmatrix}$$

   Solution:

   This $\boxed{\text{IS}}$ a linear transformation. Let's check the properties:

   (1) $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$: Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^2$. Then, we can write them as
   $$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$
   By definition, we have that
   $$T(\vec{x} + \vec{y}) = T \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + x_2 + y_2 \\ x_2 + y_2 \end{bmatrix}$$
   and
   $$\begin{aligned} T(\vec{x}) + T(\vec{y}) &= T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + T \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 + x_2 + y_1 + y_2 \\ x_2 + y_2 \end{bmatrix} \end{aligned}$$
   Thus, we see that $T(\vec{x}+\vec{y}) = T(\vec{x})+T(\vec{y})$, so this property holds.

   (2) $T(c\vec{x}) = cT(\vec{x})$: Let $\vec{x}$ be as above, and let c be a scalar. Then,
   $$T(c\vec{x}) = T \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix} = \begin{bmatrix} cx_1 + cx_2 \\ cx_2 \end{bmatrix}$$
   while
   $$cT(\vec{x}) = c \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 + cx_2 \\ cx_2 \end{bmatrix}$$
   Therefore, $T(c\vec{x}) = cT(\vec{x})$, so this property holds as well.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$, with

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$$

Solution:

This is $\boxed{\text{NOT}}$ a linear transformation. It can be checked that neither property (1) nor property (2) from above hold. Let's show that property (2) doesn't hold. Let

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and let $c = 2$. Then,

$$T(\vec{x}) = T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and therefore, we have that

$$2T(\vec{x}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

However, we have

$$T(2\vec{x}) = T \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Thus, we see that $2T(\vec{x}) \neq T(2\vec{x})$, and hence T is not a linear transformation.

(c) Fix an $m \times n$ matrix A. Then, let $T : \mathcal{M}_{lm} \to \mathcal{M}_{ln}$, with

$$T(B) = BA$$

Solution:

This $\boxed{\text{IS}}$ a linear transformation. Let's check the properties:

(1) $T(B + C) = T(B) + T(C)$: By definition, we have that

$$T(B + C) = (B + C)A = BA + CA$$

since matrix multiplication distributes. Also, we have that

$$T(B) + T(C) = BA + CA$$

by definition. Thus, we see that $T(B + C) = T(B) + T(C)$, so this property holds.

(2) $T(dB) = dT(B)$: By definition,

$$T(dB) = (dB)A = dBA$$

while

$$dT(B) = dBA$$

Therefore, $T(dB) = dT(B)$, so this property holds as well.

(d) Let V be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$, under normal function addition and scalar multiplication. Then, let $T : V \to \mathbb{R}^2$, with

$$T(f) = \begin{bmatrix} f(0) \\ f(1)+1 \end{bmatrix}$$

Solution:

This is $\boxed{\text{NOT}}$ a linear transformation. Neither property (1) nor property (2) hold. Let's show that property (1) doesn't hold. Let f and g be functions in V such that $f(x) = 1$, $g(x) = x$. Then, we have that

$$(f+g)(x) = x+1$$

Therefore, we see that

$$T(f) + T(g) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

while

$$T(f+g) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Thus, $T(f) + T(g) \neq T(f+g)$, and therefore T is not a linear transformation.

2. For the following linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$, find a matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$,

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ 3y \\ 4x+5y \end{bmatrix}$$

Solution:

To figure out the matrix for a linear transformation from $\mathbb{R}^n$, we find the matrix A whose first column is $T(\vec{e}_1)$, whose second column is $T(\vec{e}_2)$ – in general, whose ith column is $T(\vec{e}_i)$. Here, by definition we have that

$$T(\vec{e}_1) = T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, T(\vec{e}_2) = T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$$

Thus,

$$\boxed{A = \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 4 & 5 \end{bmatrix}}$$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$, satisfying

$$T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, T \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Solution:

We need to find $T(\vec{e}_2)$ and $T(\vec{e}_2)$. Given the information we have, this is easiest to do by writing $\vec{e}_1$ and $\vec{e}_2$ as linear combinations of

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

We start with $\vec{e}_1$. We solve

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Setting up the system of equations as usual and solving yields $c_1 = 3, c_2 = -1$. Thus, we have that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Now, since $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$, and $T(c\vec{x}) = cT(\vec{x})$, this gets us that

$$\begin{aligned} T \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= T \left(3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) \\ &= T \left(3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) + T \left((-1) \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) \\ &= 3T \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1)T \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= 3 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + (-1) \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -11 \end{bmatrix} \end{aligned}$$

Similarly, we get that

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = (-2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

and a calculation like the one above yields

$$\begin{aligned} T \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= (-2)T \begin{bmatrix} 1 \\ 1 \end{bmatrix} + T \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= (-2) \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 \\ 9 \end{bmatrix} \end{aligned}$$

Combining the information, we see that

$$\boxed{A = \begin{bmatrix} 5 & -4 \\ -11 & 9 \end{bmatrix}}$$

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$, where $T(\vec{x})$ is $\vec{x}$ rotated by $30°$ clockwise.

Solution:

Again, we need to figure out $T(\vec{e}_1)$ and $T(\vec{e}_2)$. Basic trigonometry shows that

$$T(\vec{e}_1) = T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix}$$
$$T(\vec{e}_1) = T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

Thus,

$$\boxed{A = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}}$$