Legendre Polynomials.

The Legendre poly^{n} $P_n(t)$ is defined to be the poly^{n} sol^{n} to

$$(1-t^2)y''-2ty'+n(n+1)y=0$$

which satisfies $P_n(1)=1$.

Let $y=\sum_{m\geq 0} a_m t^m$,

$$y'=\sum_{m\geq 1} m a_m t^{m-1}=\sum_{m\geq 0}(m+1)a_{m+1}t^m$$

$$y''=\sum_{m\geq 1} m(m+1)a_{m+1}t^{m-1}=\sum_{m\geq 0}(m+1)(m+2)a_{m+2}t^m$$

$$=\sum_{m\geq 0}(m-1)m a_m t^{m-2}$$

$$\Rightarrow 0=\sum_{m\geq 0}(m+1)(m+2)a_{m+2}t^m$$

$$-\sum_{m\geq 0}(m-1)m a_m t^m$$

$$-\sum_{m\geq 0}2m a_m t^m+\sum_{m\geq 0}n(n+1)a_m t^m$$

$$=\sum_{m\geq 0}\left[(m+1)(m+2)a_{m+2}+a_m\left(n(n+1)-m^2+m-2m\right)\right]t^m$$

$$=\sum_{m\geq 0}\left[(m+1)(m+2)a_{m+2}+\left(n(n+1)-m(m+1)\right)a_m\right]t^m$$

$$\therefore a_{m+2} = \frac{m(m+1) - n(n+1)}{(m+1)(m+2)} a_m$$

At $m = n$ the sequence terminates. So for a poly$^{\underline{n}}$ sol$^{\underline{n}}$ require that for $m \not\equiv n \mod 2$, require $a_m = 0$.

<u>$n = 0$:</u> $a_{m+2} = \frac{m}{m+2} a_m$

$$\Rightarrow a_0 \neq 0,\ a_1 = 0,\ \boxed{P_0(t) = 1}$$

<u>$n = 1$:</u> $a_{even} = 0$.

$$a_3 = \frac{2-2}{6} a_1 = 0. \quad \boxed{\therefore P_1(t) = t}$$

<u>$n = 2$:</u> $a_{odd} = 0$.

$$a_2 = \frac{-6}{2} a_0 = -3a_0,\ a_4 = 0$$

$$P_2(t) = a_0 - 3a_0 t^2 = a_0(1 - 3t^2)$$

$$1 = P_2(1) = -2a_0 \qquad \therefore P_2(t) = \frac{-1}{2}(1 - 3t^2)$$

$\underline{n=3}$: $a_{even} = 0$

$$a_3 = \frac{2-12}{6}a_1 = \frac{-10}{6}a_1 = -\frac{5}{3}a_1$$

$$a_5 = 0$$

$$P_3(t) = a_1\left(t - \frac{5}{3}t^3\right)$$

$$1 = P_3(1) = a_1\left(1 - \frac{5}{3}\right) = -\frac{2}{3}a_1$$

$$\boxed{\begin{aligned} \therefore P_3(t) &= -\frac{3}{2}\left(t - \frac{5}{3}t^3\right) \\ &= \left(\frac{5}{2}t^2 - \frac{3}{2}\right)t \end{aligned}}$$