TOPICS IN D-MODULES.

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1. Introduction: Local Systems.

1.1. Local Systems. Consider the differential equation (DE)
\[
\frac{d}{dz} u(z) = \frac{\lambda}{z} u(z), \quad \lambda \in \mathbb{C} \text{ a constant.}
\]
An undergraduate might say that a solution to this is \( z^\lambda \) – but what does this mean for \( \lambda \) non-integral? 
\[
u(z) = z^\lambda = e^{\lambda \log(z)} \text{ for a branch of log.}
\]

Figure 1. Analytic continuation of a solution of a DE around a loop.

The analytic continuation changes
\[
\begin{align*}
\log(z) & \mapsto \log(z) + 2\pi i \\
z^\lambda & \mapsto e^{2\pi i \lambda} z^\lambda
\end{align*}
\]
“monodromy”

Could say that \( z^\lambda \) is a multivalued function. The space of local solutions form a local system. We will see various definitions of local systems shortly.

Let \( X \) be a (reasonable) topological space, and \( \text{Op}(X) \) the lattice of open sets; i.e. a category with a unique morphism \( U \to V \) if \( U \subset V \).

Definition 1. A presheaf (of vector spaces) \( \mathcal{F} \) is a functor
\[
\mathcal{F} : \text{Op}(X)^{\text{op}} \to \text{Vect} = \text{vector spaces over } \mathbb{C}.
\]
I.e. for every \( U \subset X \) open, \( \mathcal{F}(U) \) is a vector space, and if \( U \subset V \) there is a unique linear map (think “restriction”) \( \mathcal{F}(V) \to \mathcal{F}(U) \). We call \( \mathcal{F}(U) = \Gamma(U; \mathcal{F}) = \text{“sections of } \mathcal{F} \text{ of } U \text{”} \).
Definition 2 (Non-precise). A presheaf $\mathcal{F}$ is a sheaf if sections of $\mathcal{F}$ on $U$ are "precisely determined by their restriction to any open cover."

Definition 3 (Formal). If $\{U_i\}$ is an open cover of $U$,

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightarrow \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equaliser diagram.

Example 1. Various kinds of functions form sheaves:

- all functions;
- $C^\infty_X$ smooth functions (if $X$ is a smooth manifold);
- $\mathcal{O}^\text{an}_X$ holomorphic functions (if $X$ is a complex manifold).

Example 2. The functor $C^\text{pre}_X$ of constant functions on $X$ is not a sheaf (consider the two point space with the discrete topology).

Example 3. We can improve the above example so that it becomes a sheaf by instead considering the locally constant functions $\mathbb{C}_X$. We call this a constant sheaf.

Remark Presheaves form a category (functor category); sheaves are contained in this as a full subcategory.

Definition 4. A sheaf is locally constant of rank $r$ if there is an open cover $\{U_i\}$ of $X$ such that

$$\mathcal{F}|_{U_i} \cong \mathbb{C}_U^{\oplus r}.$$ 

Example 4. The solutions to the DE at the start of this section form a locally constant sheaf. On contractible sets not containing 0 the solutions form a 1d vector space, and there are no global sections.

Definition 5 (One definition of a local system). A local system is a locally constant sheaf.

Definition 6. The stalk of a sheaf $\mathcal{F}$ at a point $x$ is

$$\mathcal{F}_x = \colim_{\text{open } U \ni x} \mathcal{F}(U).$$

We sometimes call elements of the stalk germs of sections of $\mathcal{F}$ near $x$.

I.e. the stalk is sections on $\mathcal{F}(U \ni x)$ where $s_U \in \mathcal{F}(U)$ and $s_V \in \mathcal{F}(V)$ are equivalent if there is $W \subset U \cap V$ containing $x$ such that $s_U|_W = s_V|_W$.

If $\mathcal{F}$ is locally constant,

$$\mathcal{F}_x = \mathcal{F}(U) \cong \mathbb{C}^r$$

for some small enough open set $U \ni x$.

Since the rank of a locally constant sheaf is constant on components, if $x$ and $y$ are in the same component, $\mathcal{F}_x \cong \mathcal{F}_y$. How can we realize this isomorphism?

If $\gamma : [0, 1] \to X$ is a path with $\gamma(0) = x$ and $\gamma(1) = y$, and $\mathcal{F}$ is a locally constant sheaf, we can make the following observation:
Proposition 1.1. There is an isomorphism $t_\gamma : F_x \sim \rightarrow F_y$.

Proof. The isomorphism $t_\gamma$ is given by the chain of isomorphisms

$$F_x \sim \rightarrow F(U_1) \sim \rightarrow F(U_1 \cap U_2) \sim \rightarrow \cdots \rightarrow F(U_{n-1} \cap U_n) \sim \rightarrow F(U_n) \sim \rightarrow F_y.$$

\[\square\]

Proposition 1.2. If $\gamma$ is homotopic to $\tilde{\gamma}$ then $t_\gamma = t_{\tilde{\gamma}}$.

Proof. Can contain both paths in a compact contractible set, then run a similar proof to above. \[\square\]

Definition 7. The fundamental groupoid of $X$, denoted $\Pi_1(X)$, is a category with

- Objects: the points of $X$,
- Morphisms: \{paths from $x$ to $y$\}/homotopy.

Observe that this really is a groupoid (not difficult).

Theorem 1.3. There is an equivalence of categories

$$\{\text{Locally constant sheaves on } X\} \sim \rightarrow \{\text{Functors } \Pi_1(X) \rightarrow \text{Vect}^{fd}\}$$

$$\mathcal{F} \longrightarrow \{ \begin{array}{c} x \rightarrow \mathcal{F}_x \\ t_\gamma : \mathcal{F}_x \rightarrow \mathcal{F}_y \end{array} \}$$

This gives us another perspective on local systems. In particular, they are a homotopic invariant, not a homeomorphism invariant.

1.1.1. Another perspective: Where do local systems come from? If $X$ is a smooth manifold and $\pi : E \rightarrow X$ is a smooth vector bundle, we can functorially produce the sheaf of sections $\mathcal{E}$:

$$\mathcal{E}(U) = \Gamma(U; E) \ni (s : U \rightarrow E : \pi \circ s = \text{id}).$$

How can we make sense of local constancy? Connections. Write $\Gamma(E) = \Gamma(X; E) = \mathcal{E}(X)$ for global sections, and let $\nabla : \Gamma(E) \rightarrow \Omega^1(E)$ be a connection.

Remark Can properly think of this as a map of sheaves $\mathcal{E} \rightarrow \Omega^1(\mathcal{E})$ – we get away with conflating bundles with sheaves because our sheaves have nice properties (in particular we have partitions of unity at our disposal).
What is the candidate for local constancy? *Horizontal sections:*

\[ \ker(\nabla) = \{ s \in \Gamma(E) | \nabla s = 0 \}. \]

When does this behave nicely? This goes back to Frobenius: for concreteness let's work locally with local coordinates \( x_1, \ldots, x_n \) on \( X \) and \( s_1, \ldots, s_r \) a basis of (local) sections of \( E \). Then

\[ \nabla \frac{\partial}{\partial x_i}(s_j) = \sum_k a_{ijk}s_k, \]

where \( a_{ijk} \) is the connection matrix. Write \( s = \sum f_j s_j \). Then

\[ \nabla s = 0 \iff \left\{ \frac{\partial f_j}{\partial x_i} + \sum_k a_{ijk}f_k = 0 \right\} \text{ system of PDEs.} \]

What then does it mean to have a locally constant sheaf? Morally: "Given an initial condition at a point, there is a unique solution to this system of PDEs on some contractible neighbourhood." Let's phrase this in a more precise and familiar way. Write \( \nabla_i := \nabla \frac{\partial}{\partial x_i} \).

**Theorem 1.4** (Frobenius Theorem). If \( \nabla_i \nabla_j = \nabla_j \nabla_i \) for all \( i, j \), then the sheaf \( \ker(\nabla) \) is locally constant. We then say that the connection is flat, or integrable.

**Remark** Globally this is phrased as \( \nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X,Y]} \) where \( X, Y \) are vector fields.

Thus we can expand on the previous theorem.

**Theorem 1.5.** There is an equivalence of categories

\[ \{ \text{Locally constant sheaves on } X \} \overset{\sim}{\leftrightarrow} \{ \text{Functors } \Pi_1(X) \to \text{Vect}^{\text{fd}} \} \overset{\sim}{\leftrightarrow} \{ \text{Integrable/flat connections on } X \} \]

2. SHEAVES.

Fix \( X \) a topological space. Recall that a presheaf on \( X \) is a functor \( \text{Op}(X)^{\text{op}} \to \text{Vect} \); or more generally we could take

\[ \mathcal{F} : \text{Op}(X)^{\text{op}} \to \left\{ \begin{array}{l} \text{Set} \\ \text{Ab} \\ \text{Rings} \\ \text{etc.} \end{array} \right\} \]

\( \mathcal{F} \) is a sheaf if "sections can be defined locally".

**Example 5.** Given \( \pi : Y \to X \) a continuous map of spaces, *sections of \( Y \) over \( X \)*, \( \mathcal{Y}_{/X} \) is a presheaf defined by

\[ \mathcal{Y}_{/X}(U) = \{ s : U \to Y | \pi s(x) = x \}. \]

This is a presheaf of *sets*.

**Claim:** \( \mathcal{Y}_{/X} \) is a sheaf.

**Proof.** Suppose \( \{ U_i \} \) is an open cover of \( U \subset X \), \( U \) open.

1. If \( s, s' \in \mathcal{Y}_{/X}(U) \) such that \( s|_{U_i} = s'|_{U_i} \) for all \( i \), then it is clear that \( s = s' \) (functions are defined pointwise).

2. If \( s_i \in \mathcal{Y}_{/X}(U_i) \) such that \( s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \) for all \( i, j \), we can define

\[ s \in \mathcal{Y}_{/X}(U) \text{ by } s(x) = s_i(x) \text{ if } x \in U_i. \]

\( \square \)
This example is very important – it gives us a huge variety of examples, and in some sense it gives us all sheaves (we will make this precise soon).

**Example 6.** If \( Y = X \times Z \xrightarrow{\pi_Y} X \),
\[
\mathcal{S}_{X \times Z/X}(U) = C(U, Z) = \{ \text{cts. functions } U \to Z \}.
\]

**Example 7.** If \( Z = \mathbb{C} \) with the Euclidean topology, \( \mathcal{S}_{X \times \mathbb{C}/X} = \mathbb{C}_X \), continuous complex valued functions on \( X \). Observe that we can consider this as a sheaf of sets, vector spaces, rings, etc.

**Example 8.** If \( Z = \mathbb{C}^{\text{disc}} \), \( \mathbb{C} \) with the discrete topology, then
\[
\mathcal{S}_{X \times \mathbb{C}^{\text{disc}}/X} = \mathbb{C}_X,
\]
locally constant functions on \( X \) (constant sheaf).

**Example 9.** If \( E \xrightarrow{\pi} X \) is a complex vector bundle, i.e. there is an open cover \( \{U_i\} \) of \( X \) such that
\[
E|_{U_i} = \pi^{-1}(U_i) \cong U_i \times \mathbb{C}^r,
\]
commuting with projection
\[
\pi^{-1}(U_i) \xrightarrow{=} U_i \xrightarrow{\pi^{-1}} \mathbb{C}^r
\]
such that
\[
\pi^{-1}(x) = E_x \xrightarrow{\sim} \{x\} \times \mathbb{C}^r.
\]
Then \( \mathcal{S}_{E/X} = \mathcal{E} \) is a sheaf of vector spaces – but in fact it is even more than that:

\( \mathcal{E} \) is a sheaf of modules for the sheaf of rings \( C_X \).

**Definition 8.** A bundle of groups with fibre/structure group \( G \) is a map \( E \to X \) of spaces such that there is an open cover \( \{U_i\} \) of \( X \) such that \( E|_{U_i} \cong U_i \times G \).

**Example 10.** What is a bundle of sets? A set is a discrete topological space, so a bundle of sets is a covering space.

**Example 11.** A bundle of topological spaces is a fibre bundle.

**Example 12.** This is slightly subtle: a bundle of discrete vector spaces is **not** a vector bundle! Sections here are somehow ‘locally constant’. This will give us another way to think about local systems.

**Claim:** If \( Y \to X \) is a covering space, then \( \mathcal{S}_{Y/X} \) is a locally constant sheaf of sets. I.e., locally it looks like the sheaf of locally constant functions.

We would like a converse to this: given a locally constant sheaf, produce a covering space. We will actually consider a more general construction.

2.1. Étale spaces. Given a sheaf \( \mathcal{F} \) on \( X \), define the étale space \( \text{Ét}(\mathcal{F}) \) as follows:

- As a set, it is the collection of germs of sections of \( \mathcal{F} \),
\[
\text{Ét}(\mathcal{F}) = \coprod_{x \in X} \mathcal{F}_x.
\]
- Topology: If \( U \subset X \) is open, \( s \in \mathcal{F}(U) \), we have \( s_x \in \mathcal{F}_x \) for all \( x \in U \). Then declare the sets \( \{s_x \mid x \in U\} \subset \text{Ét}(\mathcal{F}) \) to be open, and \( \{s_x \mid x \in U\} \leftrightarrow U \) a homeomorphism.
We have a map \( \bar{\text{Et}}(\mathcal{F}) \to X \) with \( \mathcal{F}_x = \pi^{-1}(x) \). In fact, \( \pi \) is a local homeomorphism. I.e. if \( s_x \in \bar{\text{Et}}(\mathcal{F}) \) then there exists \( U \ni s_x \) open in \( \bar{\text{Et}}(\mathcal{F}) \) such that \( \pi|_U \) is a homeomorphism onto its image.

**Claim:** \( \mathcal{I}_{\bar{\text{Et}}(\mathcal{F})/X} \cong \mathcal{F} \).

For each \( x \in X \), need to give an element of \( \mathcal{F}_x \) (to define a section \( \bar{\text{Et}}(\mathcal{F}) \to X \)). Then we want to show that given the defined topology, the stalks glue to a legitimate section of \( \mathcal{F} \).

A little clearer: check that the assignment

\[
\mathcal{F} \to \mathcal{I}_{\bar{\text{Et}}(\mathcal{F})/X}
\]

\[
s \in \mathcal{F}(U) \mapsto \{ x \mapsto s_x \}
\]

is continuous. So: this gives an equivalence of categories

\[
\text{Local homeomorphisms over } X \leftrightarrow \text{Sheaves of sets on } X
\]

Inside of this, we have the equivalence

\[
\text{Local homeomorphisms over } X \leftrightarrow \text{Sheaves of sets on } X
\]

\[
\text{Covering spaces} \leftrightarrow \text{Locally constant sheaves}
\]

**Remark** If \( \mathcal{F} \) is a presheaf, \( \bar{\text{Et}}(\mathcal{F}) \to X \) is still a local homeomorphism, so we can still take its sheaf of sections

\[
\mathcal{F}^{\text{sh}} = \mathcal{F}^+ = \text{sh}(\mathcal{F}) := \mathcal{I}_{\bar{\text{Et}}(\mathcal{F})/X}
\]

which we call the *sheafification* of \( \mathcal{F} \). Sheafification is left adjoint to the inclusion functor \( i : \text{Sheaves}(X) \to \text{Presheaves}(X) \),

\[
\text{Hom}_{\text{Presheaves}(X)}(\mathcal{F}, i(\mathcal{G})) \cong \text{Hom}_{\text{Sheaves}(X)}(\mathcal{F}^{\text{sh}}, \mathcal{G}).
\]

### 2.2. Functors on sheaves.

Given \( f : X \to Y \), what can we do with sheaves? We would like to be able to push them forward and pull them back:

\[
\text{Sh}(X) \xrightarrow{f_*} \text{Sh}(Y)
\]

\[
\text{Sh}(X) \xleftarrow{f^*} \text{Sh}(Y)
\]

Given a sheaf \( \mathcal{F} \) on \( X \), define

\[
f_*(\mathcal{F})(U) = \mathcal{F}(f \in (U))
\]

where \( U \subset Y \) is open. If \( \mathcal{G} \) is a sheaf on \( Y \), define

\[
(f^* \mathcal{G})^{\text{pre}}(V) = \colim_{U \subset f(V)} \mathcal{G}(U),
\]

where \( V \) runs over the open sets containing \( U \), and define

\[
f^* \mathcal{G} = \text{sh}(f^* \mathcal{G})^{\text{pre}}.
\]

In terms of local homeomorphisms (étalé spaces), the pullback sheaf/inverse image sheaf should be the sections of the pullback

\[
\begin{array}{ccc}
X \times_Y E & \longrightarrow & E \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
\]

**Example 13.** Let \( f : X \to \text{pt} \), \( \mathcal{F} \) a sheaf on \( X \). Then \( f_*(\mathcal{F}) = \Gamma(\mathcal{F}) = \mathcal{F}(X) \).
So $f_*$ is a generalization of global sections. Think of $f_*$ as being sections along the fibres (at least when $f$ looks like a fibration).

If $A$ is a set (i.e. a sheaf on a point), then

\[ f^*(A) = A_X, \] the constant sheaf.

**Example 14.** Let $i : \{x\} \hookrightarrow X$. Then $i^* \mathcal{F} = \mathcal{F}_x$.

So we can think of pullback as a generalization of the stalk, at least when we are looking at inclusion of a subspace.

**Definition 9.** If $A$ is a set, $i_* A$ is called the **skyscraper sheaf** at $x$.

### 2.3. Homological properties of sheaves

From now on we will turn away from sheaves of sets. Today, $\text{Sh}(X)$ means sheaves of *abelian groups* on $X$. We are interested in the following fact:

“$\text{Sh}(X)$ is an abelian category.”

**Definition 10.** A category $\mathcal{C}$ is **abelian** if

1. it contains a zero object (initial and terminal),
2. it contains all binary products and coproducts,
3. it contains all kernels and cokernels,
4. every monomorphism is a kernel and every epimorphism is a cokernel.

**Example 15.** The category of all groups is a non-example – if $H \subset G$ is non-normal then $G/H$ is not a group.

**Properties:** In an abelian category,\n
- $\text{Hom}_\mathcal{C}(A,B)$ is an abelian group.
- Finite products = finite coproducts.
- The first isomorphism theorem holds:

\[
\begin{array}{ccc}
\ker(f) & \longrightarrow & A \\
\downarrow & & \downarrow f \\
\coim(f) & \sim & \text{im}(f) \\
\end{array}
\]

\[
\begin{array}{ccc}
& & B \\
\downarrow & & \downarrow g \\
& & C \\
\end{array}
\]

**Example 16.** For a ring $R$, $R$-modules form an abelian category.

**Definition 11.** In an abelian category $\mathcal{C}$,

\[
A \xrightarrow{f} B \xrightarrow{g} C
\]

is called **exact** if $\text{im}(f) = \ker(g)$.

Observe that this implies that $g \circ f = 0$.

**Definition 12.** A **short exact sequence** (SES) is an exact sequence of the form

\[
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0.
\]

**Example 17.** $0 \rightarrow A \rightarrow A \oplus C \rightarrow C \rightarrow 0$; such a SES is called **split**.

**Proposition 2.1** (Splitting lemma). A SES $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is split iff either

1. there exists some $s : B \rightarrow A$ such that $s \circ f = id_A$; or,
2. there exists some $t : C \rightarrow B$ such that $g \circ t = id_C$. 

**Warning!** This is a property of *abelian* categories. A $t$-splitting in the category of all groups would only tell us that $B$ is a *semi*-direct product.

**Definition 13.** A complex in $\mathcal{C}$ is a sequence of objects and morphisms
\[
\cdots \to A^{i-1} \xrightarrow{d_{i-1}} A^i \xrightarrow{d_i} A^{i+1} \to \cdots = A^*\]
such that $d_i \circ d_{i-1} = 0$ for any $i$. We often write $d^2 = 0$. Then the cohomology of this complex is
\[
H^i(A^*) = \frac{\ker(d_i)}{\text{im} (d_{i-1})}.
\]

**Definition 14.** A morphism of complexes $A^* \to B^*$ is a commutative diagram
\[
\cdots \xrightarrow{} A^i \xrightarrow{} A^{i+1} \xrightarrow{} \cdots \\
\downarrow \quad \quad \quad \quad \downarrow \\
\cdots \xrightarrow{} B^i \xrightarrow{} B^{i+1} \xrightarrow{} \cdots 
\]
A morphism $A^* \to B^*$ is a quasi-isomorphism if it induces an isomorphism of cohomology $H^i(A^*) \xrightarrow{\sim} H^i(B^*)$.

If $\mathcal{C}$ and $\mathcal{D}$ are abelian categories and $F : \mathcal{C} \to \mathcal{D}$ is a functor, we say
- $F$ is additive if it preserves finite coproducts.
- $F$ is left exact if it is additive and preserves kernels.
- $F$ is right exact if it is additive and preserves cokernels.

If $0 \to A \to B \to C \to 0$ is a SES in $\mathcal{C}$, then
- If $F$ is left exact, $0 \to F(A) \to F(B) \to F(C)$ is exact.
- If $F$ is right exact, $F(A) \to F(B) \to F(C) \to 0$ is exact.

In abelian groups, we have the functors (for fixed $A \in \text{Ab}$)
\[
\text{Hom}(A, -) : \text{Ab} \to \text{Ab} \quad \text{(left exact)} \\
A \otimes_{\mathbb{Z}} (-) : \text{Ab} \to \text{Ab} \quad \text{(right exact)}
\]

**Example 18.** Let $A = \mathbb{Z}/2$ and take the SES $0 \to \mathbb{Z} \xrightarrow{x} \mathbb{Z} \to \mathbb{Z}/2 \to 0$.
- Apply $\text{Hom}(A, -) : 0 \to 0 \to 0 \to \mathbb{Z}/2$.
- Apply $A \otimes_{\mathbb{Z}} (-) : \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$.

**Claim:** $\text{Sh}(X)$ is an abelian category.

**Sketch of proof.** Consider $\mathcal{F}, \mathcal{G} \in \text{Sh}(X), \phi : \mathcal{F} \to \mathcal{G}$.
\[
\ker(\phi)(U) = \ker(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)),
\]
so $\ker(\phi) \in \text{Sh}(X)$. We can define
\[
coker(\phi) := \text{sh}(\text{coker}(\phi)^{\text{pre}})
\]
where
\[
coker(\phi)^{\text{pre}}(U) = \text{coker}(\phi_U) = \mathcal{G}/\phi_U(\mathcal{F}(U)).
\]

**Example 19 (Cokernel presheaf is not a sheaf.)** Let $X = \mathbb{R}$, $\mathcal{F} = \mathbb{Z}_x$, $\mathcal{G} = \mathbb{Z}_x \oplus \mathbb{Z}_y$, $x \neq y$ in $\mathbb{R}$. How can we define a map $\phi : \mathbb{Z}_x \to \mathbb{Z}_x \oplus \mathbb{Z}_y$? Such a map is equivalent to a global section $s \in \Gamma(\mathbb{Z}_x \oplus \mathbb{Z}_y) \cong \mathbb{Z} \oplus \mathbb{Z}$. Choose $(1, 1) \in \mathbb{Z} \oplus \mathbb{Z}$. What is the cokernel of this map?
Figure 3. Cover of $\mathbb{R}$ by two open sets $U$ and $V$.

Then

$$\phi_U : \mathbb{Z} \xrightarrow{\sim} (\mathbb{Z}_x \oplus \mathbb{Z}_y)(U) = \mathbb{Z},$$

$$\phi_V : \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}.$$

So

$$\text{coker}(\phi)^{\text{pre}}(U) = 0 \text{ and } \text{coker}(\phi)^{\text{pre}}(V) = 0.$$

But!

$$\text{coker}(\phi)^{\text{pre}}(\mathbb{R}) = \text{coker}(\mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z} \oplus \mathbb{Z}) \cong \mathbb{Z}.$$

Recall that sheafification is global sections of the étale space $\mathcal{E}(\mathcal{F}) = \coprod \mathcal{F}_z$. From the above, $\mathcal{F}_z = 0$ for all $z \in \mathbb{R}$. Thus,

$$\text{coker}(\phi) = 0.$$

Example 20. Let $X = \mathbb{R}/t$, $\mathcal{F} = \mathcal{G} = \mathcal{C}_\infty^\infty$ (complex valued smooth functions). We have a map

$$\frac{d}{dt} : \mathcal{C}_\infty \rightarrow \mathcal{C}_\infty^\infty,$$

with $\text{ker} \left( \frac{d}{dt} \right) = \mathbb{C}_\mathbb{R}$ (locally constant functions).

What about the cokernel? Let $U \subset \mathbb{R}$ be open, $f \in \mathcal{C}_\infty(U)$. Want to construct a function $F$ such that $\frac{d}{dt}F = f$; we can do this (fundamental theorem of calculus), e.g. let $F(t) = \int_a^t f(x)dx$. So this map of sheaves is surjective.

Example 21. Let $X = S^1 = \mathbb{R}/\mathbb{Z}$, $\mathcal{F} = \mathcal{G} = \mathcal{C}_S^\infty$. Then $\text{ker} \left( \frac{d}{dt} \right) = \mathbb{C}_{S^1}$ again. Since any open interval $U \subset S^1$ is diffeomorphic to $\mathbb{R}$, $\frac{d}{dt}|_U : \mathcal{C}_U^\infty \rightarrow \mathcal{C}_U^\infty$ is surjective. So the cokernel sheaf is coker $\left( \frac{d}{dt} \right) = 0$.

But: That the constant function $1_{S^1} \in \mathcal{C}_{S^1}^\infty$. This is not in the image of $\frac{d}{dt}|_{S^1} : \mathcal{C}_\infty(S^1) \rightarrow \mathcal{C}_\infty^\infty(S^1)$.

Example 22. Phrased differently, we have a SES of sheaves on $S^1$,

$$0 \rightarrow \mathbb{C}_{S^1} \rightarrow \mathcal{C}^\infty_{S^1} \xrightarrow{\frac{d}{dt}} \mathcal{C}^\infty_{S^1} \rightarrow 0.$$

But if we take global sections $\Gamma$,

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{C}^\infty(S^1) \xrightarrow{\frac{d}{dt}} \mathcal{C}^\infty(S^1),$$

i.e. the final map is not surjective. This is a manifestation of the fact the $\Gamma$ is left exact (but not exact). If we do take the cokernel we have

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{C}^\infty(S^1) \xrightarrow{\frac{d}{dt}} \mathcal{C}^\infty(S^1) \rightarrow H_{\text{dR}}^1(S^1) = \mathbb{C}.$$

Proposition 2.2. In general, $\Gamma : \text{Sh}(X) \rightarrow \text{Ab}$ is left exact.

On the other hand, exactness can be checked locally.

Proposition 2.3. A sequence of sheaves $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ is exact if and only if $\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$ is exact for all $x \in X$. 
If \( f : X \to Y \) is a map of topological spaces, \( \mathcal{F} \in \text{Sh}(X) \) and \( \mathcal{G} \in \text{Sh}(Y) \), recall we have \( f_*(\mathcal{F}) \in \text{Sh}(Y) \) and \( f^*(\mathcal{G}) \in \text{Sh}(X) \).

Recall that
\[
f^*(\mathcal{G})(U) = \text{sh} \left( U \mapsto \lim_{V \supset f(U)} \mathcal{G}(V) \right).
\]

**Proposition 2.4.** \( f^* \) is left adjoint to \( f_* \).

I.e. \( \text{Hom}_{\text{Sh}(X)}(f^* \mathcal{G}, \mathcal{F}) \cong \text{Hom}_{\text{Sh}(Y)}(\mathcal{G}, f_* \mathcal{F}) \) is a natural bijection.

**Proof.** Want to construct natural transformations
\[
f^*f_* \xrightarrow{\epsilon} \text{id}_{\text{Sh}(X)} \quad \text{and} \quad \text{id}_{\text{Sh}(Y)} \xrightarrow{\eta} f_*f^*.
\]

Why? Given \( u, c \) as above,
\[
\text{Hom}(f^*\mathcal{G}, \mathcal{F}) \xrightarrow{f^*} \text{Hom}(f_*f^*\mathcal{G}, f_*\mathcal{F}) \xrightarrow{\epsilon} \text{Hom}(\mathcal{G}, f_*\mathcal{F})
\]

2.4. Simplicial homology. Simplices:

![Figure 5. Low dimensional simplices.](image)
So,

\[ n\text{-simplex} \leftrightarrow (0 \to 1 \to \cdots \to n) = [n]. \]

The faces of an \( n \)-simplex are the ordered subsets \( S \subset \{0, \ldots, n\} \).

**Figure 6.** Faces of a 2-simplex.

Define the *simplex category* \( \Delta \):

- Objects: \([0], [1], [2], \ldots\)
- Morphisms: order preserving maps \([n] \to [m]\).

**Definition 15.** A *simplicial set* is a functor \( X : \Delta^\text{op} \to \text{Set} \) (i.e. a presheaf on \( \Delta \)).

So we have the category \( s\text{Set} \), and there is an embedding

\[
\Delta \xrightarrow{\text{Yoneda}} \ s\text{Set}
\]

\([n] \longmapsto \text{Hom}(\dashv, [n]) =: [n]

The fully faithful category given by this is the one with all simplices and all colimits of such (things glued together from simplices).

A simplicial set gives a recipe for building a simplicial topological space.

\[
s\text{Set} \xrightarrow{|\cdot|} \text{Top}
\]

\([n] \longmapsto \Delta^n
\]

where \(|\cdot|\) is *geometric realization* and we extend this definition to all \( s\text{Set} \) preserving colimits. If \( X : \Delta^\text{op} \to \text{Set} \) is a simplicial set we write

\[
X([n]) = X_n
\]

for the set of \( n \)-simplices, and if we have a map \( f : [n] \to [m] \) in \( \Delta \) this induces \( X(f) : X_m \to X_n \) and \( f_\Delta : \Delta^m \to \Delta^n \) in \( \text{Set} \). So now take concretely

\[
|X| = \prod_n (X_n \times \Delta^n) / \sim
\]

where

\[
(\sigma, f_\Delta(t)) \sim (X(f)(\sigma), t) \quad \text{for } \sigma \in X_n, t \in \Delta^n.
\]

Consider the following diagram in \( \Delta \):

\[
[0] \xleftarrow{s^0} \xrightarrow{s^n} [1] \xrightarrow{d^i} [2] \xrightarrow{d^i} [3] \cdots
\]
The arrows in this diagram give all possible order preserving maps between adjacent simplices. The maps shown are distinguished maps which we call coface \((d^i)\) and codegeneracy \((s^i)\) maps. If \(X\) is a sSet we have a diagram with face and degeneracy maps

\[
\begin{array}{c}
X_0 \\ \Longleftrightarrow \\ X_1 \\ \Longleftrightarrow \\ X_2 \\
\end{array}
\]

A simplex in \(X_n\) is called degenerate if it is in the image of a degeneracy map. Think:

“Degenerate simplices are secretly lower-dimensional simplices.”

**Figure 7.** Simplicial description of \(S^1\) for example 23.

**Example 23.** How to prescribe the complex from Figure 7? As a semi-simplical set (use only face maps),

\[
X_0 = \{a\} \quad \{b\} = X_1,
\]

so that we send both faces to the point \(a\). As a simplical set we would need to keep track of the degeneracies, e.g. \(X_1 = \{b, s_0(a)\}\).

If \(Y\) is a topological space, we can produce a simplicial set \(S(Y)\) by talking

\[
S(Y)_n = \text{Hom}_{\text{Top}}(\Delta^n, Y),
\]

the singular simplicial set. This is a sSet, since

\[
f : [m] \to [n] \quad \text{induces} \quad f_\Delta : \Delta^m \to \Delta^n \quad \text{induces} \quad S(Y)_n \to S(Y)_m.
\]

**Theorem 2.5.** If \(Y\) is a CW-complex, then \(|S(Y)| \simeq Y\) (homotopy equivalence).

We can also talk about simplicial abelian groups,

\[
\Delta^{op} \to \text{Ab},
\]

and more generally simplicial objects in a category \(\mathcal{C}\) are \(\Delta^{op} \to \mathcal{C}\).

**Example 24.** If \(X\) is a simplicial set we can define a “free” simplicial abelian group \(\mathbb{Z}X\) by

\[
(\mathbb{Z}X)_n = \mathbb{Z} \cdot X_n.
\]

Given a simplicial abelian group \(A\) we can make a chain complex \(C(A)_*\):

\[
C(A)_n = A_n, \quad \delta_n : A_n \to A_{n-1} \quad \text{via} \quad \sum_i (-1)^i d_i(a),
\]

and one can prove that \(\delta_{n-1} \delta_n = 0\) (left as an exercise).

**Remark** This construction did not use the degeneracy maps – so this makes sense for semi-simplicial sets.

We have a subcomplex (check!) of degenerate simplices \(D(A) \subset C(A)\).
**Proposition 2.6.** $D(A)$ is chain homotopic to 0.

In fact,

$$C(A) = D(A) \oplus N(A)$$

where $N(A)$ is the normalised chain complex and $N(A) \sim C(A)$ (chain homotopic).

If $X$ is a simplicial set we can define

$$C_\bullet(X; \mathbb{Z}) = C(\mathbb{Z}X),$$

and the singular homology is

$$H_i(X; \mathbb{Z}) = H_i(C(\mathbb{Z}X)).$$

If $Y$ is a topological space then $S(Y)$ produces

$$C_i^{\text{sing}}(Y; \mathbb{Z})$$ (singular chains on $Y$), and $H_i^{\text{sing}}(Y; \mathbb{Z})$ (singular homology).

**Figure 8.** Nondegenerate simplices of $S^1$.

**Example 25.** Claim that

$$N(S^1; \mathbb{Z}) = \mathbb{Z}a \leftarrow \mathbb{Z}b$$

degree:

$$\begin{array}{ccc}
0 & 1 \\
\end{array}$$

Of course,

$$D(\mathbb{Z}S^1) = \mathbb{Z}a \leftarrow \mathbb{Z}b \oplus \mathbb{Z} s_0(a) \leftarrow \cdots$$

plus higher degenerate simplices which make no contribution to homology.

We can also define more generally $H_i(X; A)$ for $A$ an abelian group.

2.4.1. **Homology with coefficients in a local system.** Suppose $\mathcal{E}$ is a local system on $|X|$ (i.e. $\mathcal{E}$ is a locally constant sheaf of abelian groups). Want to define $H_i(X; \mathcal{E})$. Take (provisionally)

$$C_n(X; \mathcal{E}) = \{(x, s) \mid x \in X_n, s \in \Gamma(x^*\mathcal{E})\}.$$

What does this mean? $x \in X_n$, so think of this as $x : \Delta^n \to |X|$. But we can’t add simplices, so we actually want our $n$-chains to be:

$$C_n(X; \mathcal{E}) = \{(x, s) \mid x \in X_n, s \in \Gamma(x^*\mathcal{E})\}$$

(i.e. the abelian group generated by the terms in the angle brackets.) Now, the $n$-simplex is contractible, thus $x^*\mathcal{E}$ is trivializable (so we can take sections):

$$x^*\mathcal{E} \xrightarrow{x} \mathcal{E}$$

Now stalkwise, $A = \mathcal{E}_y$ (for some $y \in |X|$), and so on $\Delta^n$, $\Gamma(x^*\mathcal{E}) = A$.

If we took the trivial local system, we would see that we recover our previous notion of singular homology.

We define face maps $d_i : C_n(X; \mathcal{E}) \to C_{n-1}(X; \mathcal{E})$ as before by restricting sections to faces.
**Example 26.** See Figure 9.

![Figure 9](image)

**Figure 9.** Visual representation of the “simplices” in local cohomology.

Then

\[ \partial := \sum (-1)^i d_i : C_n(X; \mathcal{E}) \to C_{n-1}(X; \mathcal{E}). \]

**Example 27.** \( X = S^1 \), so \( \pi_1(S^1) = \mathbb{Z} \). Then

Local system on \( S^1 \) \( \iff \) Rep of \( \pi_1(S^1) \) \( \iff \) \( (A, t \in \text{Aut}(A)) \)

where \( A \) is the stalk of \( \mathcal{E} \) at some chosen basepoint.

![Figure 10](image)

**Figure 10.** A local system on \( S^1 \).

What is our complex?

\[
\begin{array}{ccc}
A & \xleftarrow{id-t} & A \\
\text{degree:} & 0 & 1
\end{array}
\]

which we can see graphically via:

![Diagram](image)

**Example 28.** If \( \mathcal{E} = \mathbb{Z}_{S^1} \), then we have \( \mathbb{Z} \xleftarrow{id} \mathbb{Z} \).

**Example 29.** If \( A = \mathbb{C}, \ t = \lambda \in \mathbb{C}^\times \), then we have

\[ \mathbb{C} \xleftarrow{id - \lambda} \mathbb{C} \]

So if \( \lambda \neq 1 \), \( \text{id - \lambda : \mathbb{C} \cong \mathbb{C} \), so \( H_0 = 1, H_1 = 0, \) etc...
2.4.2. Cohomology with coefficients in a local system. Above, we have defined for a sSet/topological space \( X \) and local system \( \mathcal{E} \)

\[
(\Delta^i \to X) \in C_i(X; \mathcal{E}) \xrightarrow{H} H_i(X; \mathcal{E})
\]

We can give this a slightly more down to earth description: an \( i \)-simplex with coefficients in \( \mathcal{E} \) is an \( i \)-simplex with a lift \( \text{Ét}(\mathcal{E}) \)

\[
\Delta^i \longrightarrow X
\]

and the differential is induced by the face inclusions \( \Delta^{i-1} \hookrightarrow \Delta^i \). We can also define cohomology with coefficients in a local system, \( H^i(X; \mathcal{E}) \), by taking the cochains to be

\[
C^i(X; \mathcal{E}) := \left\{ \phi \mid \phi \text{ assigns to each simplex a lift, } \sigma \xrightarrow{\phi} \text{Ét}(\mathcal{E}) \right\}
\]

and defining the differential \( d : C^i \to C^{i+1} \) to be the alternating sum of coface maps \( d^r : C^i \to C^{i+1} \), where

\[
d^r(\phi)(\sigma^{i+1}) = \phi(d_r(\sigma^{i+1})).
\]

### 3. Sheaf cohomology as a derived functor.

#### 3.1. Idea and motivation. Where are we going? Our next goal is to prove Poincaré Duality: If \( M \) is a closed \( n \)-manifold, then

\[
H_i(M; \mathcal{O}_M) \cong H^{n-i}(M; \mathcal{O}_M),
\]

where \( \mathcal{O}_M \) is the orientation local system.

**Remark** If \( M \) is orientable, \( \mathcal{O}_M \) is the constant sheaf.

**Example 30.** If \( M = S^1 \),

\[
\begin{align*}
H_0, H_1 &= \mathbb{Z} & \text{Poincaré Duality} \\
H^1, H^0 &= \mathbb{Z}
\end{align*}
\]

We will actually recover Poincaré duality as a special case of Verdier duality. So, in order to continue, we need to define sheaf cohomology.

**Motivating sheaf cohomology.** If \( \mathcal{E} \) is a local system, what is \( H^0(X; \mathcal{E}) \)? A 0-cochain assigns to each 0-simplex \( x \in X \) an element of the stalk at \( x \). I.e. this is a discontinuous section of \( \text{Ét}(\mathcal{E}) \).

**What does it mean to be closed?** If I have two points and a path between them, the elements of the stalk have to be compatible. I.e. we must have a continuous function:
I.e. $H^0(X; \mathcal{E}) = \Gamma(\mathcal{E})$.

In general, given a sheaf $\mathcal{F}$ we want to define functors $H^i(X; \mathcal{F})$ such that $H^0(X; \mathcal{F}) = \Gamma(\mathcal{F})$.

**Idea:** These functors should measure the *non-exactness* of $\Gamma$.

Given a SES of sheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, taking $\Gamma$ gives

$$0 \to \Gamma(\mathcal{F}') \to \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}'') \to ?$$

In order to measure the failure of exactness, the we will define a next term in this sequence called $H^1(\mathcal{F}')$ – it turns out that this will only depend on $H^1(\mathcal{F}')$.

**Definition 16.** A sheaf $\mathcal{F}$ is called *flabby* (or *flasque*) if for each $V \subseteq U \subseteq X$ of open sets, $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

**Proposition 3.1.** Suppose $0 \to \mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}'' \to 0$ is a SES, and $\mathcal{F}'$ is flabby. Then

$$0 \to \Gamma(\mathcal{F}') \to \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}'') \to 0$$

is exact.

**Proof.** Let $s'' \in \Gamma(\mathcal{F}'') = \mathcal{F}''(X)$. Let’s define a set

$$\mathcal{S} := \{(U, s) \mid s \in \mathcal{F}(U) \text{ such that } g(s) = s''|_U\}. $$

We want to show that there is an element of this set with $U = X$. $\mathcal{S}$ has a partial order

$$(U_1, s_1) \leq (U_2, s_2) \text{ if } U_1 \subseteq U_2 \text{ and } s_2|_{U_1} = s_1.$$  

Zorn’s lemma implies that there is a maximal $(U, s)$ in $\mathcal{S}$.

Suppose that $x \in X - U$. We can find $V \ni x$ open in $X$, and $t \in \mathcal{F}(V)$ such that $g(t) = s''|_V$. 

![Figure 11. $H^0$ gives continuous sections.](image)
On $U \cap V$, $q := s|_{U \cap V} - t|_{U \cap V} \in \mathcal{F}(U \cap V)$ has the property that $g(q) = 0$. By left exactness, $g(q) = 0$ implies that there exists $w \in \mathcal{F}'(U \cap V)$ such that $f(w) = q$.

Now, since $\mathcal{F}$ is flabby, there exists $r \in \mathcal{F}'(X)$ such that $r|_{U \cap V} = w$. Let $t' = t + f(r)|_V \in \mathcal{F}(V)$. Note that

$$g(t') = s'|_V = g(t)$$

by exactness of $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, and that $s|_{U \cap V} = t'|_{U \cap V}$. Thus there exists a section $\tilde{s} \in \mathcal{F}(U \cup V)$ such that $g(\tilde{s}) = s'|_{U \cup V}$. But this contradicts maximality of $(U,s)$. Thus, $U = X$. □

3.2. **Homological Algebra.** Idea: If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a SES, we want a (functorial) LES

$$
0 \longrightarrow \Gamma(\mathcal{F}') \longrightarrow \Gamma(\mathcal{F}) \longrightarrow \Gamma(\mathcal{F}'')
$$

$$
\downarrow \quad H^1(\mathcal{F}') \quad \downarrow \quad H^1(\mathcal{F}) \quad \downarrow \quad H^1(\mathcal{F}'')
$$

$$
\downarrow \quad H^2(\mathcal{F}') \quad \downarrow \quad H^2(\mathcal{F}) \quad \downarrow \quad H^2(\mathcal{F}'') \quad \longrightarrow \cdots
$$

The collection of functors $\{H^i\}_{i \in \mathbb{Z}}$ is called a $\delta$-functor.
There are also naturality conditions. Given a map of SES,

\[
\begin{array}{ccc}
\mathcal{F}''_1 & \longrightarrow & \mathcal{F}''_2 \\
\uparrow & & \uparrow \\
\mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 \\
\uparrow & & \uparrow \\
\mathcal{F}'_1 & \longrightarrow & \mathcal{F}'_2 \\
\uparrow & & \uparrow \\
0 & \longrightarrow & 0
\end{array}
\]

require that

\[
\begin{array}{ccc}
H^{i+1}(\mathcal{F}'_1) & \longrightarrow & H^{i+1}(\mathcal{F}'_2) \\
\delta & & \delta \\
\downarrow & & \downarrow \\
H^{i+1}(\mathcal{F}''_1) & \longrightarrow & H^{i+1}(\mathcal{F}''_2)
\end{array}
\]

Where does this come from? If \(0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0\) is a SES of cochain complexes (concentrated in nonnegative degrees), there exists a LES

\[
\begin{array}{ccccccccc}
0 & \cdots & \longrightarrow & H^i(A) & \longrightarrow & H^i(B) & \longrightarrow & H^i(C) & \longrightarrow & \delta \\
& & & \downarrow & & \downarrow & & \downarrow & & \\
& & & H^{i+1}(A) & \longrightarrow & H^{i+1}(B) & \longrightarrow & H^{i+1}(C) & \longrightarrow & \cdots
\end{array}
\]

I.e. \(\{H^i\}\) is a \(\delta\)-functor.

Suppose we have

\[
\begin{array}{ccccccc}
0 & \longrightarrow & A^i & \xrightarrow{f} & B^i & \xrightarrow{g} & C^i & \longrightarrow & 0 \\
& & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & A^{i+1} & \xrightarrow{f} & B^{i+1} & \xrightarrow{g} & C^{i+1} & \longrightarrow & 0
\end{array}
\]

We want to define a map \(H^i(C) \xrightarrow{\tilde{h}} H^{i+1}(A)\). Let \(c \in H^i(C)\), and choose a representative \(\tilde{c} \in C^i\) such that \(d\tilde{c} = 0\). There exists \(\tilde{b} \in B^i\) such that \(g(\tilde{b}) = \tilde{c}; g(d\tilde{b}) = 0\), so there exists \(\tilde{a} \in A^{i+1}\) such that \(f(\tilde{a}) = d\tilde{b}\). But now, \(f(d\tilde{a}) = 0\), so \(d\tilde{a} = 0\) and thus \(\tilde{a}\) represents a class \(a \in H^{i+1}(A)\). Thus we define

\[
h(c) = a.
\]

We can see the argument diagrammatically as follows:

\[
\begin{array}{cccccc}
\tilde{b} & \longrightarrow & \tilde{c} \\
0 & \longrightarrow & A^i & \xrightarrow{f} & B^i & \xrightarrow{g} & C^i & \longrightarrow & 0 \\
& & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & A^{i+1} & \xrightarrow{f} & B^{i+1} & \xrightarrow{g} & C^{i+1} & \longrightarrow & 0 \\
\tilde{a} & \longrightarrow & d\tilde{b}
\end{array}
\]

and so

\[
\begin{array}{cccccc}
0 & \longrightarrow & A^{i+2} & \xrightarrow{f} & B^{i+2} \\
& & \downarrow d \tilde{a} & & \downarrow 0
\end{array}
\]

Computing cohomology. Let’s assume that we’ve already constructed

\(H^i(X; -) : \text{Sh}(X) \rightarrow \text{Ab}\)

as a \(\delta\)-functor. How would we compute this?

Remark There is a category of cohomological \(\delta\)-functors, and there is a notion of a universal \(\delta\)-functor: a terminal object in this category. The \(H^i(X; -)\) will be universal in this sense.
Suppose also that there exists a collection of sheaves $\mathcal{A} := \{\mathcal{A}\}$, such that $H^i(\mathcal{A}) = 0$ for $i > 0$ when $\mathcal{A} \in \mathbb{A}$ (acyclics), and for each $\mathcal{F}$ there is some $\mathcal{A} \in \mathbb{A}$ such that $\mathcal{F} \hookrightarrow \mathcal{A}$.

It turns out that flabby sheaves are such an example:

$$\mathcal{F} \hookrightarrow G(\mathcal{F}) := \prod_{x \in X} i_x(\mathcal{F}_x).$$

$G(\mathcal{F})$ is called the sheaf of discontinuous sections of $\mathcal{F}$.

Now, let’s compute $H^1$. There is a SES

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{A}^0 \rightarrow \mathcal{K}^1 \rightarrow 0$$

which we can continue into an exact sequence by finding an acyclic $\mathcal{A}^1$ such that $\mathcal{K}^1 \hookrightarrow \mathcal{A}^1$, splicing in the result, and then repeating the procedure for $\mathcal{K}^2$ and etc.:

Taking the cohomology LES gives

$$H^0(\mathcal{F}) \rightarrow H^0(\mathcal{A}^0) \rightarrow H^0(\mathcal{K}^1) \rightarrow H^1(\mathcal{F}) \rightarrow H^1(\mathcal{A}^0) = 0,$$

since $\mathcal{A}^0$ is acyclic. So we can express

$$H^1(\mathcal{F}) = H^0(\mathcal{A}^0)/\text{im } (H^0(\mathcal{A}^0)).$$

Continuing the LES,

$$H^1(\mathcal{A}^0) = 0 \rightarrow H^1(\mathcal{K}^1) \hookrightarrow H^2(\mathcal{F}) \rightarrow 0.$$

Now as in the above splicing picture, we can play the same game for $\mathcal{K}^1 \hookrightarrow \mathcal{A}^1$:

$$H^2(\mathcal{F}) \cong H^1(\mathcal{K}^1) = H^0(\mathcal{K}^2)/\text{im } (H^0(\mathcal{A}^1)).$$

**How can we splice this information together?** $\mathcal{A}^\bullet$ is a complex, and we have a cohomology sheaf $\mathcal{H}^i(\mathcal{A}^\bullet)$. We also (importantly!) have the

$$H^i(\Gamma(\mathcal{A}^\bullet)) = H^i(\mathcal{F}).$$

Why? Exactness gives that

$$\ker(\mathcal{A}^1 \rightarrow \mathcal{A}^2) = \mathcal{K},$$

so

$$H^1(\mathcal{F}) = \ker(H^0(\mathcal{A}^1) \rightarrow H^0(\mathcal{A}^2))/\text{im } (H^0(\mathcal{A}^0) \rightarrow H^0(\mathcal{A}^1)).$$

**Summary:** If for all $\mathcal{F} \in \text{Sh}(X)$ there exists $\mathcal{A}^\bullet$ such that $\mathcal{F} \hookrightarrow \mathcal{A}^\bullet$ and $H^i(\mathcal{A}^\bullet) = 0$ for all $i > 0$, then we can compute $H^i(\mathcal{F})$ as $H^i(\Gamma(\mathcal{A}^\bullet))$. We call such an $\mathcal{A}^\bullet$ an acyclic resolution.

**Definition 17.** A $\delta$-functor which has the above property (i.e. existence of acyclic resolutions) is called effacable.

**Theorem 3.2.** An effacable $\delta$-functor is universal.
3.2.1. *Injective resolutions*. Let $C$ be an abelian category. An object $I \in C$ is *injective* if $\text{Hom}(-, I)$ is exact. I.e.

\[
0 \to A \to B \to I \to 0
\]

**Remark** If $0 \to I \to B \to C \to 0$ is a SES it is split, since

\[
0 \to I \to B \to C \to 0
\]

**Remark** If $F$ is an additive functor, it preserves split exact sequences.

**Example 31.** In $\text{Ab}$, $\mathbb{Z}$ is not injective; e.g.

\[
0 \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0,
\]

but $\mathbb{Z} \not\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. On the other hand, $\mathbb{Q}$ is injective.

**Definition 18.** A category $C$ is said to have *enough injectives* if for all $A \in C$ there exists $I$ injective such that $A \to I$. Having enough injectives implies the existence of *injective resolutions* $A \to I^\bullet$.

Suppose $F : C \to D$ is left exact, and $C$ has enough injectives. Then we can define a (universal) $\delta$-functor $R^i F$ as follows:

\[
R^i F(A) := H^i(F(I^\bullet)),
\]

where $A \to I^\bullet$ is an injective resolution.

**Why is this well defined, and why is it a $\delta$-functor?** This boils down to the *comparison lemma*

\[
A \to B \to \text{splitting map} \to I^\bullet \to J^\bullet
\]

and the *Horseshoe lemma*

\[
0 \to A \to B \to C \to 0 \to 0
\]

In particular:

\[
H^i(X; -) = R^i \Gamma(X; -).
\]

**Remark** The LES sequence of the $\delta$-functor is exactly the cohomology LES of

\[
0 \to F(I^\bullet) \to F(J^\bullet) \to F(K^\bullet) \to 0.
\]

We can compute cohomology using injective resolutions.

3.3. **Why does $\text{Sh}(X)$ have enough injectives?** $\text{Ab}$ has enough injectives, since $\mathbb{Q}/\mathbb{Z}$ is injective and

\[
A \to \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \to \prod \mathbb{Q}/\mathbb{Z}.
\]

Then for $\mathcal{F} \in \text{Sh}(X)$, we can construct injectives $I(-)$ using the above procedure (see [W]) to obtain

\[
\mathcal{F} \to G(\mathcal{F}) = \prod_{x \in X} (i_x)_*(\mathcal{F}_x) = \prod_{x \in X} (i_x)_* I(\mathcal{F}_x).
\]
3.4. Computing sheaf cohomology. Recall that we can compute cohomology using acyclic resolutions.

**Proposition 3.3.** Flabby sheaves are acyclic.

**Proof.** If \( \mathcal{F} \) is flabby, take an injective resolution \( 0 \to \mathcal{F} \to I^\bullet \). Now, injective sheaves are flabby (exercise), and in the SES

\[
0 \to \mathcal{F} \to I^\bullet \to \mathcal{K} \to 0
\]

\( \mathcal{K} \) is also flabby (exercise). So the LES of a \( \delta \)-functor gives

\[
\begin{array}{cccccc}
0 & \to & H^0(\mathcal{F}) & \to & H^0(I) & \to & H^0(\mathcal{K}) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & H^1(\mathcal{F}) & \to & 0 & \to & 0 & \to & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & H^2(\mathcal{F}) & \to & 0 & \to & \cdots & & \\
\end{array}
\]

We have \( \mathcal{F} \to G(\mathcal{F}) \), so we have from this the Godement resolution \( \mathcal{F} \to G^\bullet(\mathcal{F}) \), and so

\( H^i(X; \mathcal{F}) = H^i(\Gamma(G^\bullet(\mathcal{F}))) \).

Computing cohomology with the Godement resolution is, however, bloody stupid. Thankfully we have already seen that all we need are acyclic resolutions.

![Figure 13. Sum of two singular chains.](image)

3.4.1. Singular cohomology. Let \( \mathcal{E}_X^{\text{sing, pre}} \) be the presheaf of singular cochains on \( X \) with coefficients in \( \mathbb{Z} \). This is **not** a sheaf!

Let \( \sigma \in C_i(X) \) be as pictured in Figure 13, and let \( \varphi \in C^i(X) \) be defined by \( \varphi(\sigma) = 1 \) and \( \varphi(\tilde{\sigma}) = 0 \) if \( \tilde{\sigma} \neq \lambda \sigma \). Then in particular, \( \varphi|_U \equiv 0 \).

So \( \mathcal{E}_X^{\text{sing, pre}} \) is not a sheaf. But for sort of a silly reason: if we define \( \sigma_1 \) and \( \sigma_2 \) as in Figure 13 then \( \varphi(\sigma_i) = 0 \), but \( \varphi(\sigma_1 + \sigma_2) = \varphi(\sigma) = 1 \). We really should have that \( \sigma_1 + \sigma_2 \) and \( \sigma \) represent the same object.

So, define \( \mathcal{E}_X^{\text{sing}} := \text{sh}(\mathcal{E}_X^{\text{sing, pre}}) \).

**Claims:**
(1) There is a quasi-isomorphism $\mathcal{C}^{\text{sing}}(X)\Rightarrow C^{\text{sing},\text{pre}}(X)$ (this uses “the lemma of small chains”).

(2) $\mathcal{C}^{\text{sing}}$ is flabby.

(3) If $X$ is locally contractible, then $\mathbb{Z}_X \rightarrow \mathcal{C}^{\text{sing}}$ is a resolution. For this, exactness on small enough contractible opens is sufficient; then $H^0 = \mathbb{Z}$ and $H^i = 0$ for $i > 0$.

So if $X$ is locally contractible, then for a locally constant sheaf $\mathcal{E}$, $H^i(X; \mathcal{E}) = H^i_{\text{sing}}(X; \mathcal{E})$.

3.4.2. De Rham cohomology. If $M$ is a smooth manifold, then we have the sheaf of smooth $i$-forms on $M$, $\mathcal{A}^i_M$. Using the de Rham differential we get a complex

$$\mathcal{A}^\bullet_M := C^\infty(M) \rightarrow \mathcal{A}^1_M \rightarrow \mathcal{A}^2_M \rightarrow \cdots$$

Then the Poincaré lemma says that $C_M \rightarrow \mathcal{A}^\bullet_M$ is a resolution. I.e.,

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{A}^0(\mathbb{R}^n) \rightarrow \mathcal{A}^1(\mathbb{R}^n) \rightarrow \cdots \rightarrow \mathcal{A}^n(\mathbb{R}^n) \rightarrow 0$$

is exact ("any closed form on $\mathbb{R}^n$ is exact").

Remark Really this is just an application of the fundamental theorem of calculus.

Now: the $\mathcal{A}^i_M$ are not flabby, but they are fine (and soft).

Exercise 3.1. Prove that the $\mathcal{A}^i_M$ are acyclic (hint: partitions of unity). Thus it will follow that $H^i(M; \mathbb{C}_M) \cong H^i(\mathcal{A}^\bullet_M(M)) = H^i_{\text{dR}}(M; \mathbb{C})$.

3.4.3. Dolbeault resolution. If $X$ is a complex manifold, we have sheaves $\mathcal{O}_X$ of holomorphic functions and $\Omega^i_X$ of holomorphic $i$-forms. We have that

$$\mathcal{C}_X \rightarrow \Omega^0_X \xrightarrow{\partial} \cdots \rightarrow \Omega^n_X$$

is a resolution. But we can’t compute cohomology with this: the $\Omega^i_X$ are not acyclic!

What we can do is the following. Begin by embedding $\Omega^0_X \hookrightarrow \mathcal{A}^0_X$ (smooth functions). Then we have a resolution

$$\Omega^0_X \rightarrow \mathcal{A}^0_X \rightarrow \mathcal{A}^{0,1}_X \rightarrow \mathcal{A}^{0,2}_X \rightarrow \cdots \rightarrow \mathcal{A}^{0,n}_X,$$

and this is an acyclic resolution. In fact, we can form a double complex:

```
\begin{array}{c}
\mathcal{A}^{0,n} \\
\mathcal{A}^{0,n-1} \rightarrow \\
\mathcal{A}^{0,n-2} \rightarrow \\
\cdots \\
\mathcal{A}^{0,1} \rightarrow \\
\mathcal{A}^{0,0} \rightarrow \\
\Omega^0_X \\
\Omega^1_X \\
\cdots \\
\Omega^n_X \\
\end{array}
```

Then we have that in fact

$$H^p(\Omega^q_X) = H^p_{\text{Dol}}(X) =: H^p(\Gamma(\mathcal{A}^q_\bullet), \overline{\partial}).$$

If \( f : X \to Y \) we have the functor
\[
f_* : \text{Sh}(X) \to \text{Sh}(Y);
\]
observe that \( f_* = \Gamma \) if \( f : X \to \text{pt} \). \( f_* \) is a left exact functor between abelian categories, so we can define the right derived functors of \( f_* \)
\[
R^i f_* (\mathcal{F}) = \mathcal{H}^i (f_* (\mathcal{I})^•)
\]
where \( \mathcal{I} \) is an injective/flabby/acyclic resolution of \( \mathcal{F} \), and so \( f_* (\mathcal{I}) \) is a complex of sheaves on \( Y \). Thus \( R^i f_* (\mathcal{F}) \in \text{Sh}(Y) \).

**Example 32.** If \( f : X \to Y \) is a fibration (or fibre bundle, or submersion), then for the constant sheaf \( R^i f_* (\mathbb{Z}_X) \in \text{Sh}(Y) \), and on stalks
\[
R^i f_* (\mathbb{Z}_X)_y = H^i (f^{-1}(y); \mathbb{Z}).
\]

**Warning!** Take \( i : \mathbb{C}^\times \hookrightarrow \mathbb{Z} \) and pushforward the constant sheaf (or sheaf of singular cochains). Then since locally around \( 0 \in \mathbb{C} \) we have punctured open balls \( (\simeq \mathbb{C}^\times) \) our stalk at zero picks up
\[
R^1 i_* (\mathbb{Z}_{\mathbb{C}^\times})_0 \simeq H^1 (\mathbb{C}^\times; \mathbb{Z}) = \mathbb{Z}.
\]

Now, if \( X \xrightarrow{f} Y \xrightarrow{g} Z \) it is easy to show that \( (g \circ f)_* = g_* f_* \). What about \( R^i g_* \circ R^i f_* \)?

**Example 33.** Using \( \mathbb{Z} = \text{pt} \) we might wish to try and compute cohomology on a fibration by understanding how the cohomology of the fibres varies and taking the sheaf cohomology of that. This is computed using the Leray spectral sequence, which is a potential topic for another day.

For the moment we take a different tack. We have
\[
R^i f_* (\mathcal{F}) = \mathcal{H}^i (f_* (\mathcal{I}^0) \to f_* (\mathcal{I}^1) \to \cdots),
\]
so we define the total derived functor to be
\[
R f_* (\mathcal{I}) = f_* (\mathcal{I}^0) \to f_* (\mathcal{I}^1) \to \cdots
\]
We will worry about the dependence upon a choice of \( \mathcal{I} \) in a second. Observe that
\[
R g_* \circ R f_* = R (g \circ f)_* = g_* f_* (\mathcal{I}^•).
\]
This makes sense, as the pushforward of an injective resolution is still an injective resolution.

**Remark** The equation \( R g_* \circ R f_* = R (g \circ f)_* \) secretly encodes the Leray-Serre spectral sequence.

**Remark** If \( \mathcal{I}^• \) is a complex of sheaves (bounded below) we can find an injective resolution
\[
\mathcal{I}^• \xrightarrow{\text{quasi-isomorphism}} \mathcal{I}^•,
\]
and \( R f_* (\mathcal{I}^•) = f_* (\mathcal{I}^•) \).

What is going on here? We would like to say that we have a functor
\[
R f_* : \{\text{Complexes of sheaves on } X.\} \to \{\text{Complexes of sheaves on } Y.\},
\]
but this doesn’t make sense – there are sheaves which are quasi-isomorphic but not isomorphic. We can fix this by taking (roughly)
\[
R f_* : D^+ (\text{Sh}(X)) \to D^+ (\text{Sh}(Y)),
\]
where \( D^+ (\text{Sh}(X)) \) is the derived category,
\[
D^+ (\text{Sh}(X)) = \{\text{bounded below complexes of sheaves on } X\}[\text{quasi-isomorphisms}]^{-1},
\]
a category whose objects are complexes, whose morphisms are morphisms of complexes, but where all quasi-isomorphisms have been inverted.

**Warning!** This is not rigorous definition – there are problems we will tackle later.
Remark If \( \mathcal{F} \) and \( \mathcal{G} \) are objects in \( D^+(\text{Sh}(X)) \) and \( \mathcal{H}^n(\mathcal{F}) \cong \mathcal{H}^n(\mathcal{G}) \) for all \( i \), it is not necessarily true that \( \mathcal{F} \cong \mathcal{G} \) (quasi-isomorphism).

Example 34. Consider the Hopf fibration \( S^1 \hookrightarrow S^3 \xrightarrow{f} S^2 \). We want to consider \( Rf_* (\mathbb{Z}_3) \in D^+ \text{Sh}(S^2) \).

Let’s look at the cohomology objects \( R^if_* (\mathbb{Z}_3) \in \text{Sh}(S^2) \). \( f \) is a fibration, so for \( U \subset S^2 \) a small disk,

\[
\Gamma(U; R^if_* (\mathbb{Z}_3)) = H^i \left( \frac{f^{-1}(U)}{S^1 \times U} \right) \cong H^i(S^1; \mathbb{Z}).
\]

So the \( R^if_* (\mathbb{Z}_3) \) are locally constant, and since \( S^2 \) is simply connected, locally constant sheaves are constant. Hence,

\[
\begin{align*}
R^0 f_* (\mathbb{Z}_3) &= \mathbb{Z}_{S^2} \quad \text{(measuring } H^0 \text{ of fibres)} \\
R^1 f_* (\mathbb{Z}_3) &= \mathbb{Z}_{S^2} \quad \text{(measuring } H^1 \text{ of fibres)}
\end{align*}
\]

What is the total derived functor \( Rf_* (\mathbb{Z}_3) \)? There is always an obvious guess:

\[
\mathbb{Z}_{S^2} \oplus \mathbb{Z}_{S^2}[-1].
\]

Consider

\[
S^3 \xrightarrow{f} S^2 \to \text{pt},
\]

which gives

\[
\begin{align*}
Rf_* &= R(p \circ f)_* = R\Gamma(S^3; \mathcal{F}) \quad \text{(where } \mathcal{F} \text{ is compactly supported cohomology)}
\end{align*}
\]

Now if this were the total derived functor, we would have

\[
Rp_* (\mathbb{Z}_{S^2} \oplus \mathbb{Z}_{S^2}[-1]) = \underbrace{Rp_* (\mathbb{Z}_{S^2})}_{C^* (S^2; \mathbb{Z})} \oplus \underbrace{Rp_* (\mathbb{Z}_{S^2})[-1]}_{C^* (S^2; \mathbb{Z})[-1]}.
\]

and upon taking cohomology of this complex, we get

\[
H^* (Rp_* (\mathbb{Z}_{S^2} \oplus \mathbb{Z}_{S^2}[-1])) = \mathbb{Z} \oplus \mathbb{Z}[-1] \oplus \mathbb{Z}[-2] \oplus \mathbb{Z}[-3].
\]

But \( H^* (S^3) = \mathbb{Z} \oplus \mathbb{Z}[-3] \), and so

\[
Rf_* (\mathbb{Z}_3) \neq \mathbb{Z}_{S^2} \oplus \mathbb{Z}_{S^2}[-1].
\]

Remark A spectral sequence calculation relates \( H^* (S^3) \) and \( H^* (Rp_* (\mathbb{Z}_{S^2} \oplus \mathbb{Z}_{S^2}[-1])) \). The calculation makes transparent how the degree 1 and 2 \( \mathbb{Z} \) terms are killed off.

4.1. Compactly supported sections. Define the **compactly supported sections** functor by

\[
\Gamma_c(X; \mathcal{F}) : \text{Sh}(X) \to \text{Ab}
\]

\[
\Gamma_c(X; \mathcal{F}) = \{ s \in \Gamma(X; \mathcal{F}) | \text{supp}(s) \subseteq K \subseteq X \text{ for some compact } K \}
\]

where

\[
\text{supp}(s) := \{ x \in X | s_x \in \mathcal{F}_x \text{ is nonzero} \}.
\]

Exercise 4.1. Show that \( \text{supp}(s) \) is closed.

\( \Gamma_c \) is left exact, so we can consider its right derived functors which we call the **compactly supported cohomology** of \( \mathcal{F} \):

\[
R^n \Gamma_c(X; \mathcal{F}) = H^n_c(X; \mathcal{F}).
\]

Example 35. If \( X \) is compact, then \( \Gamma_c(X; \mathcal{F}) = \Gamma(X; \mathcal{F}) \).

Example 36. If \( X = \mathbb{R}^n \), \( \Gamma_c(X; \mathbb{Z}_{\mathbb{R}^n}) = 0 \).
Given \( f : X \to Y \) we can define

\[ f_! : \text{Sh}(X) \to \text{Sh}(Y) \]

by

\[ f!(\mathcal{F})(V) = \left\{ s \in \mathcal{F}(f^{-1}(V)) \left| \begin{array}{c} \text{supp}(s) \subseteq f^{-1}(V), \text{ and} \\ \text{supp}(s) \overset{f|_{\text{supp}(s)}}{\to} V \text{ is proper} \end{array} \right. \right\}. \]

(Recall that a map is proper if the preimage of a compact set is compact.)

We should assume some ‘niceness’ properties – e.g. Hausdorff, etc. We would need to change our notion of properness to work with, e.g. the Zariski topology for a variety.

**Example 37.** If \( Y \) is a point, \( f_! = \Gamma_c \)

Observe that

a) If \( i : Z \hookrightarrow X \) is a closed embedding, then \( i \) is proper.

b) If \( j : U \hookrightarrow X \) is an open embedding, then \( j \) is not proper.

**Example 38.**

\[
\begin{array}{ccc}
\mathbb{C}^\times & \xrightarrow{\text{open emb.}} & \mathbb{C} \\
\downarrow & & \downarrow \\
D^\times & \xrightarrow{\text{open emb.}} & D
\end{array}
\]

**4.2. Summary of induced functors so far.** A map \( f : X \to Y \) induces:

\[
\begin{array}{ccc}
f_* : \text{Sh}(X) & \xleftarrow{\text{right adjoint}} & \text{Sh}(Y) : f^* \\
\uparrow & & \uparrow \\
Rf_* : D^+\text{Sh}(X) & \xleftarrow{\text{right adjoint}} & D^+\text{Sh}(Y) : f^* \\
\uparrow & & \uparrow \\
f_! : \text{Sh}(X) & \longrightarrow & \text{Sh}(Y) \\
\downarrow & & \downarrow \\
Rf_! : D^+\text{Sh}(X) & \longrightarrow & D^+(\text{Sh}(Y))
\end{array}
\]

We call \( f_! \) the pushforward/direct image with proper supports.

**Example 39.** If \( Y = \text{pt} \), then

- \( Rf_* = R\Gamma(X; -) \).
- \( R^nf_* = H^n(X; -) \).
- \( R^nf_! = H^*_c(X; -) \).

**4.3. Computing derived functors.** Let \( j : U \hookrightarrow X \) be an open embedding, and let \( \mathcal{G} \in \text{Sh}(U) \). Then

\[
j_!(\mathcal{G})(V) = \{ s \in \Gamma(U \cap V; \mathcal{G}) \mid \text{supp}(s) \hookrightarrow V \text{ is proper} \}.
\]
Observe that $\text{supp}(s) \hookrightarrow V$ is proper iff $\text{supp}(s)$ is closed in $V$.

Let’s try and compute the stalk of this sheaf. For $x \in U$ we can always find $V' \subset U$ containing $x$; thus the condition $\text{supp}(s) \hookrightarrow V'$ is proper is vacuous (since $s \in \Gamma(U \cap V'; \mathcal{G}) = \Gamma(V'; \mathcal{G})$). Thus the stalk at $x$ is just $\mathcal{G}_x$, the stalk of $\mathcal{G}$ at $x$.

Now consider $\tilde{x} \in U$, $X \ni \tilde{x}$.

$\Gamma(V; j_! \mathcal{G}) = \{ s \in \mathcal{G}(V \cap U) \mid \text{supp}(s) \hookrightarrow V \text{ is closed} \}$.

Given such an $s$ we can take $\tilde{x} \in V' \subseteq V$ such that $V' \cap \text{supp}(s) = \emptyset$ (this probably requires our space to be Hausdorff).

Then $s \mapsto 0 \in (j_! \mathcal{G})_{\tilde{x}}$.

We summarize:

$$(j_! \mathcal{G})_x = \begin{cases} 
\mathcal{G}_x & \text{if } x \in U, \\
0 & \text{if } x \notin U.
\end{cases}$$

We call this extension of $\mathcal{G}$ by 0.

Now, it is clear that $j^* j_! \mathcal{G} \simeq \mathcal{G}$. What about $j_! j^* \mathcal{F}$? There is a map $j_! j^* \mathcal{F} \rightarrow \mathcal{F}$ given on open sets $V$ by taking a section on $\mathcal{F}(U \cap V)$ and extending by 0 to all of $V$.

**Proposition 4.1.** $j_!$ is left adjoint to $j^*$. 

*Figure 14. Defining and computing !-pushforward.*

*Figure 15. Separating $V'$ and supp$(s)$.*
The counit of the adjunction is $j_! j^* \mathcal{F} \to \mathcal{F}$, and the unit of the adjunction is $\mathcal{G} \to j^* j_! \mathcal{G}$. I.e. we have

$$\text{Hom}(j_! \mathcal{G}, \mathcal{F}) \simeq \text{Hom}(\mathcal{G}, j^* j_! \mathcal{F}).$$

To move between the unit/counit and Hom description of the adjunction, take e.g.

$$\text{Hom}(j_! G, j_! G) \simeq \text{Hom}(G, j^* j_! G).$$

Remark The functor $j_!$ is exact (since it is exact on stalks by our earlier calculation).

Remark This adjunction and exactness is only for open embeddings!

4.4. Functoriality. Consider

$$X \xrightarrow{f} Y \xleftarrow{p} \text{pt}$$

We want to compute $H^*(X; f^* \mathcal{G})$. We have

$$R\Gamma(X; f^* \mathcal{G}) \xrightarrow{Rf_* \circ Rf_*(f^* \mathcal{G})} Rf_! \circ Rf_!(f^* \mathcal{G}) \xrightarrow{\text{use adj.}} Rf_!(\mathcal{G}) \xrightarrow{R\Gamma(Y; \mathcal{G})}$$

So for example, if $\mathcal{G} = Z_Y$, then we have a map

$$H^*(Y; \mathbb{Z}) \to H^*(X; \mathbb{Z}).$$

What about for compactly supported cohomology? We can try and do the same trick. Assume $f$ is proper. Then $Rf_* \simeq Rf_!$, so we can run the same argument as above to get maps

$$H^c_*(X; \mathcal{G}) \to H^c_*(X; f^* \mathcal{G}),$$

which are given by

$$R\Gamma_c(X; f^* \mathcal{G}) \xrightarrow{Rp_! \circ Rf_!(f^* \mathcal{G})} Rf_! \circ Rf_!(f^* \mathcal{G}) \xrightarrow{\text{proper}} Rf_! \circ Rf_!(f^* \mathcal{G}) \xrightarrow{Rf_!(\mathcal{G})} R\Gamma(Y; \mathcal{G}).$$

Example 40. Consider the open embedding

$$\mathbb{C}^\times \xleftarrow{j} \mathbb{C} \subset \mathbb{R}^2 - \{0\} \subset \mathbb{R}^2$$

We understand $j_!(\mathbb{Z}_{\mathbb{C}^\times})$ – it is constant away from 0, and has no sections on sets containing 0. Now we claim

$$j_* \mathbb{Z}_{\mathbb{C}^\times} = \mathbb{Z}.$$

Letting $U$ be a small ball around 0. Then this follows from

$$(j_* \mathbb{Z}_{\mathbb{C}^\times})(U) = \mathbb{Z}_{\mathbb{C}^\times}(U \cap \mathbb{C}^\times) \cong \mathbb{Z}.$$ 

$j_*$ is not exact, so let’s compute its derived functor:

$$Rj_*(\mathbb{Z}_{\mathbb{C}^\times}) = j_*(\mathbb{C}^\times_{\mathbb{C}^\times}^{\text{sing}})(U) = C^\text{sing}(U \cap \mathbb{C}^\times).$$

$U \cap \mathbb{C}^\times \simeq S^1$, so taking cohomology gives $\mathbb{Z}$ in degrees 0 and 1. Now, we have the cohomology sheaves

$$R^0 j_*(\mathbb{Z}_{\mathbb{C}^\times}) = \mathbb{Z}, \quad R^1 j_*(\mathbb{Z}_{\mathbb{C}^\times}) = 0.$$
where we observe that we have a skyscraper sheaf at 0 since there is no first cohomology on contractible sets not containing 0.

4.5. **Open-closed decomposition.** Now, consider \( j : U \to X \leftarrow Z : i \) where \( Z = X - U \).

**Exercise 4.2.** The sequence

\[
0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0
\]

is exact. (Hint: It is easy to check exactness on stalks.)

**Think:** “The sheaf \( \mathcal{F} \) is built up from its restriction to open and closed complementary subsets.”

Apply \( R\Gamma_c(X; -) \) to the above SES:

\[
\begin{array}{cccccc}
\text{ SES } & R\Gamma_c(X; j_! j^* \mathcal{F}) & \longrightarrow & R\Gamma_c(X; \mathcal{F}) & \longrightarrow & R\Gamma_c(X; i_* i^* \mathcal{F}) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\text{ SES } & R(p_U)_!(j^* \mathcal{F}) & \longrightarrow & R(p_X)_! j_! j^* \mathcal{F} & \longrightarrow & R(p_X)_! i_* i^* \mathcal{F} & \longrightarrow & R(p_Z)_! (i^* \mathcal{F})
\end{array}
\]

where

\[
\begin{array}{ccc}
U & \xrightarrow{j} & X \\
p_U & \downarrow & \downarrow p_X \\
\text{pt} & & \text{pt}
\end{array}
\quad
\begin{array}{ccc}
X & \xleftarrow{i} & Z \\
p_X & \downarrow & \downarrow p_Z \\
\text{pt} & & \text{pt}
\end{array}
\]

Then we obtain a LES

\[
H^*_c(U; j^* \mathcal{F}) \to H^*_c(X; \mathcal{F}) \to H^*_c(Z; i^* \mathcal{F}),
\]

which we call the **LES in compactly supported (CS) cohomology.**

**Example 41.** \( j : \mathbb{R}^n \hookrightarrow S^n \leftarrow \text{pt} : i \), with \( \mathcal{F} = \mathbb{Z}_{S^n} \). Since \( \text{pt} \) and \( S^n \) are compact (and we assume \( n \geq 1 \)), we find that the LES in CS cohomology for \( \mathbb{Z}_{S^n} \) is:

\[
\begin{align*}
H^0_c(\mathbb{R}^n; \mathbb{Z}) & \to H^0(\mathbb{R}^n; \mathbb{Z}) \cong \mathbb{Z} & \to 0 \\
H^n_c(\mathbb{R}^n; \mathbb{Z}) & \to H^n(\mathbb{R}^n; \mathbb{Z}) \cong \mathbb{Z} & \to 0 \\
0 & \to 0 & \to 0 \\
\vdots & \vdots & \vdots \\
0 & \to 0 & \to 0 \\
H^n_c(\mathbb{R}^n; \mathbb{Z}) & \to H^n(\mathbb{R}^n; \mathbb{Z}) \cong \mathbb{Z} & \to H^0(\text{pt} \mathbb{Z}) \cong \mathbb{Z}
\end{align*}
\]

So

\[
H^*_c(\mathbb{R}^n; \mathbb{Z}) = \begin{cases} 
\mathbb{Z}, & \text{ if } * = n, \\
0 & \text{ else.}
\end{cases}
\]

In particular, observe that \( H^*_c \) is not a homotopy invariant – it can distinguish between \( \mathbb{R}^n \) of different dimensions.

**Notation:** If \( j : U \to X \), write \( \mathcal{F}_U \) for \( j_! j^* \mathcal{F} \).

If \( X = U_1 \cup U_2 \) with both \( U_i \) open, we have a SES of sheaves

\[
0 \to \mathcal{F}_{U_1 \cup U_2} \xrightarrow{(+, +)} \mathcal{F}_{U_1} \oplus \mathcal{F}_{U_2} \xrightarrow{(+, -)} 0.
\]

This gives rise to another LES, the **Mayer-Vietoris sequence in CS cohomology.**
Next, given \( j : U \hookrightarrow X \leftrightarrow Z : j \), consider the exact sequence

\[
0 \rightarrow \Gamma_Z \mathcal{F} \rightarrow \mathcal{F} \rightarrow j_* j^* \mathcal{F} =: \Gamma_U \mathcal{F},
\]

where

\[
(\Gamma_Z \mathcal{F})(V) = \{ s \in \mathcal{F}(V) | \text{supp}(s) \subseteq Z \}.
\]

\( \Gamma_Z \) is always left exact, but is not exact in general. So we can derive

\[
R\Gamma_Z : D^+ \text{Sh}(X) \rightarrow D^+ \text{Sh}(X).
\]

There is another functor

\[
R \Gamma_Z(X, -) : D^+ \text{Sh}(X) \rightarrow D^+ (\text{Ab}),
\]

which we call the local cohomology.

5. Verdier Duality.

Start with a map of topological spaces \( f : X \rightarrow Y \). So far we have seen the following solid arrows:

\[
\begin{align*}
D^+ \text{Sh}(X) & \xrightarrow{f^*} D^+ \text{Sh}(Y) \\
Rf_* & \xrightarrow{Rf^*} Rf_!
\end{align*}
\]

**Natural question:** Does the dashed adjoint exist?

**Example 42.** If \( U \hookrightarrow X \) is an open embedding, we have seen that \( j! \dashv j^* \dashv Rj_* \), (\( \dashv \) means “is left adjoint to”). So in this case \( j! = j^* \).

**Example 43.** If \( i : Z \hookrightarrow X \) is a closed embedding,

\[
i^* \dashv i_* \simeq i^! \dashv i^* \Gamma_Z,
\]

where \( i_* \simeq i^! \) since \( i \) is proper, \( i^* \) is exact, and \( \Gamma_Z \) is left exact. Thus we can right derive to get

\[
i^! = i^* R\Gamma_Z.
\]

**Remark** We have seen that previously the functors \( f^* \), \( Rf_! \), etc on \( D^+(\mathcal{A}) \) are induced from functors on \( \mathcal{A} \) by (right) deriving. In general, \( f^! \) is only defined in the derived category – it is not derived from the original categories.

For now, let us simplify by considering sheaves of rational vector spaces, \( D^+ \text{Sh}_\mathbb{Q}(-) \). (This is a simplification since all \( \mathbb{Q} \) vector spaces are injective.)

**In general:** We want \( Rf_! \dashv f^! \).

**Remark** If \( f \) is proper, \( Rf_* \simeq Rf_! \).

5.1. Dualizing complex. Consider \( p : X \rightarrow \text{pt} \). What is the “dualizing complex” \( p^!(\mathbb{Q}) = \omega_X^\bullet \)?

We haven’t constructed it yet, but let us deduce some of its properties. Unless otherwise stated, \( \text{Hom}_X = \text{Hom}_{D^+(\text{Sh}(Z))} \). We will have

\[
R\text{Hom}_X(\mathcal{F}, \omega_X^\bullet) = R\text{Hom}_X(\mathcal{F}, p^! \mathbb{Q}) \simeq R\text{Hom}_\mathbb{Q}(Rf_!(\mathcal{F}), \mathbb{Q}) = R\Gamma_c(X; \mathcal{F})^\vee,
\]

where we are thinking of \( \mathbb{Q} \) as a complex concentrated in degree 0, and \( -^\vee \) denotes the dual complex of vector spaces.
Remark $R\text{Hom}(A^\bullet, B^\bullet)$ means the cochain complex $\text{Hom}^\bullet(A^\bullet, I^\bullet)$ where $I^\bullet$ is an injective resolution of $B^\bullet$. I.e.

$$\text{Hom}^n(A^\bullet, I^\bullet) = \bigoplus_i \text{Hom}(A_i, I^{n+i}),$$

which we can visualize as

$$\begin{array}{c}
A^{-1} \to B^{-1} \\
| \quad | \quad | \\
A^0 \to B^0 \\
| \quad | \quad | \\
A^1 \to B^1 \\
| \quad | \quad | \\
A^2 \to B^2 \\
| \quad | \quad | \\
\vdots \quad \quad \vdots
\end{array}$$

In the above, orange arrows are degree 0 homs, blue arrows are degree 2 homs, and the black arrows denote the differential we can put on this graded vector space such that

$$H^0(\text{Hom}^\bullet(A^\bullet, I^\bullet)) = \text{Hom}(A^\bullet, I^\bullet).$$

Remark The $R\text{Hom}$ adjunction should follow from the $\text{Hom}$ adjunction by the universal property of $R$.

Example 44. Let $\mathcal{F} = \mathbb{Q}_X = p^*\mathbb{Q}$ where $p : X \to \text{pt}$. Then

$$R\text{Hom}_X(\mathbb{Q}_X; \omega_X^\bullet) = R\text{Hom}_Q(\mathbb{Q}, R\Gamma(X; \omega_X^\bullet)) = R\Gamma(X; \omega_X^\bullet)$$

by the $p^*$ adjunction, and

$$R\text{Hom}_X(\mathbb{Q}_X; \omega_X^\bullet) = R\Gamma_c(X; \mathbb{Q})^\vee = C_c^\bullet(X; \mathbb{Q})^\vee$$

by the (desired) $p!$ adjunction. Be aware that $C_c^\bullet$ is a cochain complex that is not necessarily bounded below.

Remark If $A^\bullet$ is a cochain complex of vector spaces, i.e.

$$\cdots \to A^i \to A^{i+1} \to A^{i+2} \to \cdots,$$

when we dualize we can either consider this as a chain complex, or we can relabel,

$$((A^\bullet)^\vee)^i = (A^{-i})^\vee,$$

so that $(A^\bullet)^\vee$ is a cochain complex. In this class, unless explicitly stated, all complexes are cochain complex.

Remark For $X$ finite dimensional, $R\Gamma(X; \omega_X^\bullet)$ is bounded above. Thus, in general, if $X$ is infinite dimensional the $!$-pushforward does not exist (since we used this to get $C_c^\bullet(X; \mathbb{Q})^\vee = R\Gamma(X; \omega_X^\bullet)$).

We define *Borel-Moore chains on $X$* to be

$$C_{BM}^\bullet(X; \mathbb{Q}) := C_c^\bullet(X; \mathbb{Q})^\vee.$$

Example 45. Let $\mathcal{F} = \mathbb{Q}_U = j_!\mathbb{Q}_U$ for $j : U \hookrightarrow X$ an open subset. Then

$$R\Gamma(U; \omega_U^\bullet) = R\text{Hom}(\mathbb{Q}_U; \omega_X^\bullet) = C_c^\bullet(U; \mathbb{Q})^\vee,$$

where $\omega_U^\bullet = \omega_X^\bullet|_U$.

We have an assignment

$$\begin{pmatrix}
\text{Open set } U \\
\text{in } X
\end{pmatrix} \mapsto \begin{pmatrix}
\text{Cochain complex} \\
C_c^\bullet(U; \mathbb{Q})^\vee
\end{pmatrix}.$$
Think of this as a presheaf in cochain complexes:

\[ U \hookrightarrow V \xrightarrow{\cdots} C^*_c(U; \mathbb{Q}) \hookrightarrow C^*_c(V; \mathbb{Q}) \]

\[ \downarrow \]

\[ C^*_c(U; \mathbb{Q})^\vee \hookrightarrow C^*_c(V; \mathbb{Q})^\vee \]

If we were working fully homotopically, we could take this as a definition. But, as written we have a problem – an object in the derived category is an equivalence class of complexes. To make sense of this, we would need a homotopical version of a sheaf (an \( \infty \)-stack).

5.1.1. **What is \( \omega_X^\bullet \)?** Let \( \mathcal{I}^\bullet \) be the Godement resolution of \( \mathbb{Q}_X \),

\[ \mathcal{I}^0 = \prod_{x \in X} \mathbb{Q}_x. \]

**Definition 19** (Tentative). \( \omega_X^i(U) = \Gamma_c(U; \mathcal{I}^{-i})^\vee \) (so the dualizing sheaf is concentrated in negative degrees – it is a not necessarily bounded below complex).

We now have the meaning/interpretation

\[ \mathrm{RHom}_X(\mathbb{Q}_X, \omega_X^\bullet) \cong \mathrm{R}\Gamma(\omega_X^\bullet) = \Gamma_c(X; \mathcal{I}^{-\bullet})^\vee \cong \mathrm{R}\Gamma_c(X; \mathbb{Q})^\vee \]

where the final isomorphism is because \( \mathcal{I} \) resolves the constant sheaf.

**We need the following assumption:** \( X \) is finite dimensional.

**Fact:** If \( X = \mathbb{R}^n \), \( \dim X = n \).

**Another fact:** Define \( \mathcal{K}^\bullet \in D^+\mathrm{Sh}(X) \) by

\[ \mathcal{K}^i := \begin{cases} 
\mathcal{I}^i & \text{if } i < n, \\
\text{im } (\mathcal{I}^{n-1} \to \mathcal{I}^n) & \text{if } i = n, \\
0 & \text{if } i > n.
\end{cases} \]

If \( \dim(X) = n \), then \( \mathcal{K}^\bullet \) is a soft resolution of \( \mathbb{Q}_X \).

**Recall:** \( \mathcal{I} \) is soft if for every \( Z \subseteq X \) closed, \( \Gamma(X; \mathcal{I}) \to \Gamma(Z; \mathcal{I}) \) is surjective. Soft sheaves are acyclic for \( \Gamma_c \).

**Upshot:** If \( X \) is finite dimensional, we can find a finite soft resolution of \( \mathbb{Q}_X \).

**Definition 20.** \( \omega_X^\bullet \in \mathrm{Sh}(X) \) is given by

\[ \omega_X^i(U) = \Gamma_c(U; \mathcal{K}^{-i})^\vee. \]

With the above definition and the finite dimensionality assumption,

\[ \omega_X^\bullet \in D^+(\mathrm{Sh}(X)). \]

Now, let’s look at homology complexes.

\[ \mathrm{pre} \mathcal{K}^i(\omega_X^\bullet)(U) = H^i(\Gamma_c(U; \mathcal{I}^\bullet))^\vee \]

\[ = H^i(C^*_c(U; \mathbb{Q})^\vee) \]

\[ = H^{-i}_c(U; \mathbb{Q})^\vee \]

where the final equality uses the rationality assumption (i.e. that we are dealing with vector spaces). In particular, on stalks we have

\[ \mathcal{H}^i(\omega_X^\bullet)_x = \colim_{U \ni x} H^{-i}_c(U; \mathbb{Q})^\vee = H^{-i}_c(U; \mathbb{Q}) \]

where the final equality holds for small enough \( U \) and nice enough \( X \).
Main point: If $X = M^n$ is a topological $n$-manifold, then
\[ \mathcal{H}^{-n}(\omega_X^*) \] is a locally constant sheaf.

Why? For charts $U \cong \mathbb{R}^n$,
\[ \mathcal{H}^{-n}(\omega_X^*)(U) = H^n(U; \mathbb{Q})^\vee = \mathbb{Q}. \]

Definition 21 (Orientation). $\mathcal{H}^{-n}(\omega_X^*) = \mathcal{O}_{rX}$, is called the orientation sheaf of $X$. We say that $X$ is orientable if $\mathcal{O}_{rX} \cong \mathbb{Q}_X$.

So: If $X = M^n$ is a compact, oriented manifold, we claim that
\[ H^i(X; \mathbb{Q}) \cong H_{n-i}(X; \mathbb{Q}), \]
and in particular
\[ b_i(X) = b_{n-i}(X) \] (equality of Betti numbers).

Summary of properties (so far):

1. $\mathcal{H}^{-i}(\omega_X^*)$ is the sheafification of
   \[ U \mapsto H^i_c(U; \mathbb{Q})^\vee = H_{BM}^i(U; \mathbb{Q}) := H^{-i}(U; \omega_X^*). \]
   The constant sheaf represents cochains; we can think of the dualizing complex $\omega_X^*$ as representing
   Borel-Moore chains, i.e. “locally finite” (e.g. singular) chains.

2. $\mathbb{R}\text{Hom}(\mathcal{F}^\bullet, \omega_X^*) \simeq R\Gamma_c(X; \mathcal{F})^\vee$.

Example 46.
\[ H_c^i(\mathbb{R}^n; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & i = n \\ 0, & \text{otherwise} \end{cases} = H_{BM}^i(\mathbb{R}^n; \mathbb{Q}). \]

For instance:

- $H_{BM}^2(\mathbb{R}^2; \mathbb{Q}) \cong \mathbb{Q}$, generated by all of $\mathbb{R}^2$, a locally finite sum of 2-simplices.
- $H_{BM}^0(\mathbb{R}^2; \mathbb{Q}) = 0$, since any point is the boundary of a ray headed out to $\infty$ (as per Figure 16).
If $X$ is a topological $n$-manifold, then
\[ \mathcal{H}^{-i}(\omega_X^\bullet) = 0 \quad \text{if } i \neq n, \]
and
\[ \mathcal{H}^{-n}(\omega_X^\bullet) \quad \text{is locally constant.} \]

Define the orientation sheaf to be
\[ \mathcal{O}_{rX} := \mathcal{H}^{-n}(\omega_X^\bullet). \]

We say that $X$ is orientable if $\mathcal{O}_{rX} \cong \mathbb{Q}^X$.

5.2. Poincaré Duality. If $X$ is orientable,
\[ H_{BM}^i(X; \mathbb{Q}) := H^{-i}(X; \omega_X^\bullet) \cong H^{-i}(X; \mathbb{Q}_X[n]) \cong H^{n-i}(X; \mathbb{Q}). \]

The isomorphism $H_{BM}^i(X; \mathbb{Q}) \cong H^{n-i}(X; \mathbb{Q})$ is called the Poincaré Duality isomorphism (PD). If $X$ is compact this recovers the (potentially) more familiar statement of Poincaré Duality, since then $H_{BM}^* = H^*$.

Remark $H_{BM}^i(U; \mathbb{Q}) = H^{-i}(U; \omega_U^\bullet) = H^{-i}(\omega_X^\bullet(U))$.

5.2.1. Why is PD true? There are two types of “sheaves” (up to homotopy),
\[ U \mapsto C^*(U; \mathbb{Q}) \sim \mathbb{Q}_X \]
\[ U \mapsto C_c^*(U; \mathbb{Q})^\vee = C_{BM}^*(U; \mathbb{Q}) \sim \omega_X^\bullet. \]

A priori, these are two different (pre)sheaves. But if they agree on a basis of open sets, they must agree everywhere.

So on a manifold we can check these agree (up to a shift) locally on $X$, via the computation on $\mathbb{R}^n$.

Example 47. $\mathbb{R}P^2$ and the Möbius strip are non-orientable. $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$, and $\mathcal{O}_{\mathbb{R}P^2}$ is the local system corresponding to the non-trivial representation of $\mathbb{Z}/2\mathbb{Z}$.

5.3. Borel-Moore homology.

Example 48. Consider the singular space shown in Figure 17.

![Figure 17](image)

**Figure 17.** The ‘figure 8’ singular space.

Away from the singular point, we have
\[ \mathcal{H}^{-i}(\omega_X^\bullet)_y = \begin{cases} \mathbb{Q}, & i = 1, \\ 0, & i \neq 1. \end{cases} \]

Locally around the singular point $x$, the space looks like the set $U$ of Figure 18.
So,
\[ \mathcal{H}^{-i}(\omega^*_x) = H^\text{BM}_i(U; \mathbb{Q}). \]
How can we compute this Borel-Moore homology? In general there is a Mayer-Vietoris sheaf SES,
\[ 0 \to \mathcal{F}_{Z_1 \cup Z_2} \to \mathcal{F}_{Z_1} \oplus \mathcal{F}_{Z_2} \to \mathcal{F}_{Z_1 \cap Z_2} \to 0. \]
So we have a SES of sheaves
\[ 0 \to \mathbb{Q}_{Z_1 \cup Z_2} \to \mathbb{Q}_{Z_1} \oplus \mathbb{Q}_{Z_2} \to \mathbb{Q}_{Z_1 \cap Z_2} \to 0. \]
Apply the functor \( RT_c : \)
\[
\begin{array}{c}
\cdots \\
H^0_c(Z_1 \cup Z_2) \longrightarrow H^0_c(Z_1) \oplus H^1_c(Z_2) \longrightarrow 0 \\
H^1_c(Z_1 \cup Z_2) \longrightarrow H^1_c(Z_1) \oplus H^1_c(Z_2) \longrightarrow H^0_c(Z_1 \cap Z_2)
\end{array}
\]
Since \( Z_i \cong \mathbb{R} \) and \( Z_1 \cap Z_2 = \text{pt} \), we have \( H^1_c(Z_i) \cong \mathbb{Q} \), \( H^0_c(Z_1 \cap Z_2) \cong \mathbb{Q} \), and \( H^0_c(Z_i) = 0 \). Hence we get a SES
\[ 0 \to \mathbb{Q} \to H^1_c(Z_1 \cup Z_2) \to \mathbb{Q}^2 \to 0, \]
and so
\[ H^1_c(U) \cong \mathbb{Q}^3. \]
We can see this \( H^\text{BM}_1 \) as generated by the three ‘V’ shaped cycles in Figure 19.
So we can now see in Figure 20 that $\mathcal{H}^i(\omega_X^\bullet)$ is somehow measuring singularities in our space.

**Exercise 5.1.** Think about the restriction maps for this example.

**Example 49.** Think of the cone with open ends, as in Figure 21.

Away from the singular point, the stalk of $\mathcal{H}^*(\omega_X^\bullet)$ is $\mathbb{Q}[2]$, since there the cone is locally a 2-manifold. Around the singular point, we have that locally the manifold looks again like all of $X$ – so here we need to calculate $H_{BM}^*(X; \mathbb{Q})$. We have

$$H_{BM}^*(X; \mathbb{Q}) = \begin{cases} 
\mathbb{Q}^2, & * = 2 \text{ (generated by the upper and lower cones)}, \\
\mathbb{Q}, & * = 1 \text{ (generated by the line in Figure 21)}.
\end{cases}$$

This is a rough geometric argument – we could also decompose and use a LES argument as we did in the previous example.
More generally, if \( f : X \rightarrow Y \), we want to define
\[
f^! : D^+ \text{Sh}(Y) \rightarrow D^+ \text{Sh}(X).
\]
If \( K \in \text{Sh}(X) \) is a soft sheaf, and \( F \in \text{Sh}(Y) \) is injective,
\[
f^!(K)(F) = \text{Hom}(f^!(K), F).
\]
Then define
\[
f^!(F) = (f^!_n(K) \rightarrow f^!_{n-1}(K) \rightarrow \cdots \rightarrow f^!_0(K)),
\]
where the first term is in degree \(-n\), the final term is in degree 0, and \( Q_X \rightarrow \mathcal{K} \) is the finite (length \( n \)) soft resolution as given before. By definition,
\[
\omega^*_X = f^!(Q) \quad \text{when } Y = \text{pt}.
\]

**Proposition 5.1.** \( R\text{Hom}(Rf_!(G), F) \simeq R\text{Hom}(G, f^!(F)) \).

**Proof.** See [KS]. \( \square \)

**Warning!** There is a condition on defining \( f^! : X \) and \( Y \) must both be finite dimensional.

**Definition 22.** \( \omega_{X/Y} := f^!(Q_Y) \).

5.3.1. Topological submersions. We saw (Poincaré Duality) that something nice happened with \( \omega_X \) for \( X \) a manifold. What is a similarly nice situation here?

**Definition 23.** \( f : X \rightarrow Y \) is a topological submersion of relative dimension \( d \) if for all \( x \in X \) there exists open \( U \subset X \) with \( x \in U \), such that
\[
U \xrightarrow{\cong} f(U) \times \mathbb{R}^d
\]
and \( f(U) \) is open in \( Y \).

**Remark** This is strictly stronger that just having manifolds for fibres.

**Example 50.** \( \mathbb{R} \xrightarrow{x^2} \mathbb{R} \) is not a topological submersion (it fails at 0).

**Proposition 5.2.** If \( f \) is a topological submersion of relative dimension \( d \), then
\[
\mathcal{H}^{-d}(\omega_{X/Y}) := \mathcal{O}r_{X/Y} \text{ is a local system},
\]
\[
\mathcal{H}^{-i}(\omega_{X/Y}) = 0 \text{ for } i \neq d.
\]

We say that \( X/Y \) is orientable iff \( \mathcal{O}r_{X/Y} \simeq \mathbb{Q}_X \). In that case,
\[
f^!(-) \simeq f^*(-)[d].
\]
In general: for any \( f \),
\[
f^!(-) \simeq f^*(-) \otimes \omega_{X/Y}.
\]

6. **The 6-functor formalism.**

We have functors \( Rf_*, Rf_!, f^*, f^!, - \otimes_{\mathbb{Q}} -, R\mathcal{H}om(-, -) \). These give rise to 4 kinds of (co)homology.

1. \( H^*(X; \mathbb{Q}) = Rp_*p^*(\mathbb{Q}) \) (ordinary cohomology)
2. \( H^*_c(X) = Rp_!p^*(\mathbb{Q}) \) (compactly supported cohomology)
(3) $H_{BM}^*(X) = R\rho_*p^!(\mathbb{Q})$ (Borel-Moore homology)

(4) $H_*^*(X) = Rp_*p^!(\mathbb{Q})$ (ordinary homology)

BM homology and ordinary homology use Verdier duality. Once way of thinking about this – Verdier duality gives a way to build a cosheaf from a sheaf, and we take homology of a cosheaf.

Unless explicitly stated we assume $X$ is locally compact and finite dimensional.

6.1. **Functoriality from adjunctions.** $f : X \to Y$ gives a unit map

$$1_{D^+(Y)} \longrightarrow Rf_*f^*,$$

$$\mathbb{Q}_Y \longrightarrow Rf_*(\mathbb{Q}(X))$$

which gives a map

$$R\Gamma(\mathbb{Q}_Y) = H^*(Y) \to H^*(X) = R(p_Y)_*Rf_*(\mathbb{Q}_X)$$

where

Similarly, there is a counit map

$$Rf_!f^! \longrightarrow 1_{D^+(Y)}$$

$$Rf_! f^!(\omega_Y) \longrightarrow \omega_Y$$

Applying $R(p_Y)_*$ gives

$$H^{-*}(X) = R(p_Y)_!(Rf_!(\omega_X)) \to R(p_Y)_!(\omega_Y) = H^{-*}(Y).$$

I.e.:

- $H^*$ is contravariant.
- $H_*$ is covariant.
- $H_+^*$:
  - For an open embedding $j : U \hookrightarrow X$, $j^! \simeq j^*$, i.e. $j^*$ is right adjoint to $j_!$, and so $H_+^*$ is covariant.
  - For a proper map $f : X \to Y$, $f_! = f_*$, and so $H_+^*$ is contravariant.
- $H_{BM}^*$ is essentially opposite to $H_+^*$.

If $f : X \to Y$ is a topological submersion of relative dimension $d$, and

$$\mathcal{O}_{rX/Y} (= \mathcal{H}^{-d}(\omega_{X/Y})) \simeq \mathbb{Q}_X,$$

then

$$f^!(-) \simeq f^*(-)[d] \quad \text{and so} \quad f^![-d] \simeq f^*.$$

Thus,

$$1_{D^+(Y)} \to Rf_*f^* \simeq Rf_*f^![-d].$$

Apply to $\omega_Y$:

$$\omega_Y \to Rf_*f^!(\omega_Y)[-d] = Rf_*(\omega_X)[-d],$$

so applying $R(p_Y)_*$, we have

$$R(p_Y)_*(\omega_Y) \to R(p_Y)_*Rf_*(\omega_X)[-d] = R(p_Y)_*Rf_*(\omega_X)[-d].$$
and so taking cohomology gives
\[ H_{BM}^*(Y) \to H_{BM}^*(X). \]

**Example 51.** If \( Y = \text{pt} \) (i.e. \( X \) is an orientable manifold),
\[ Q = H_{BM}^0(\text{pt}) \to H_{BM}^d(X) \quad \text{(we are using homological grading here)} \]
\[ 1 \to [X] \quad \text{(the fundamental class)} \]

So we should think of this map as relating to some relative fundamental class.

Note that for and open-closed decomposition \( j : U \hookrightarrow X \leftarrow Z = X - U : i, U \text{ open} \), we have an exact sequence of functors
\[ 0 \to j_!j^* \to j_!i^* \to 0 \]
on the level of abelian categories; so exact sequences of sheaves for any \( \mathcal{F} \in \text{Sh}(X) \)
\[ 0 \to j_!j^*(\mathcal{F}) \xrightarrow{\text{counit}} \mathcal{F} \xrightarrow{\text{unit}} i_!i^*(\mathcal{F}) \to 0. \]

Now apply \( R(p_{X!})(- \to \text{get a SES of complexes, thus a LES in compactly supported cohomology.} \)

**Example 52.** If \( \mathcal{F} = \mathbb{Q}_X \) we have
\[ \xrightarrow{\text{counit}} \bigg\}
\[ H_c^{*+1}(U) \to \cdots \]
\[ H_c^*(U) \to H_c^*(X) \to H_c^*(Z) \]

This gives rise to a distinguished triangle in \( D(X) \),
\[ Rj_!j^* \to 1_{D(X)} \to Rj_!i^* \to \cdots \]

Note that if we take \( R(p_X)_* \), we have a different interpretation,
\[ H^*(X,Z) \xrightarrow{\text{relative cohomology}} H^*(X) \to H^*(Z). \]

So:

- \( H^* \) of the decomposition deals with relative cohomology.
- \( H_c^*(X) \) is built up out of \( H_c^*(U) \) and \( H_c^*(Z) \).

We also have another distinguished triangle,
\[ Ri_*i^!(\mathcal{F}) \to \mathcal{F} \to Rj_*j^!(\mathcal{F}) \to \cdots \]
which gives rise to a LES in Borel-Moore homology
\[ H_{BM}^*(Z) \to H_{BM}^*(X) \to H_{BM}^*(U). \]

**Definition 24.** The Euler characteristic of \( X \) is
\[ \chi(X) = \sum (-1)^i \dim(H_c^i(X)). \]
Note that using \( H_c^* \) means that by the \( H_c^* \) LES,
\[ \chi(X) = \chi(U) + \chi(Z), \]
i.e. additivity of the Euler characteristic under the open-closed decomposition \( U \hookrightarrow X \leftarrow Z \).

**Proposition 6.1.** If \( i : Z \hookrightarrow X \) is a closed submanifold of codimension \( d \), and the normal bundle \( N_{Z/X} \) is orientable, then
\[ i^!(\mathbb{Q}_X) \cong \mathbb{Q}_Z[-d]. \]
Proof. Let $N$ be a tubular neighbourhood of $Z$ in $X$ as in Figure 22. Then

\[
\begin{array}{ccc}
Z & \xrightarrow{i_0} & N \\
\nearrow_i & & \searrow_{\pi} \\
& N_{Z/X} & \\
\end{array}
\]

$\pi$ is a topological submersion which in orientable. So $i^!(\mathbb{Q}_X) = i_0^!(\mathbb{Q}_N)$, and we have reduced to working on a vector bundle.

\[
i_0^!(\mathbb{Q}_N) = i_0^!(\pi_*\mathbb{Q}_Z) = i_0^!\pi^!\mathbb{Q}_Z[-d] = \mathbb{Q}_Z[-d]
\]

since $i_0^!\pi^! = id$, as $i_0$ is a section of $\pi$. Thus,

\[
i^!(\mathbb{Q}_X) = \mathbb{Q}_Z[-d].
\]

This leads to the following:

\[
i^!i^!\mathbb{Q}_X \longrightarrow \mathbb{Q}_X \\
i_*\mathbb{Q}_Z[-d] \longrightarrow \mathbb{Q}_X
\]

Now apply $Rp_*$ to this to get a Gysin map (or wrong way map)

\[H^{*-d}(Z) \rightarrow H^*(X).\]

But we have the distinguished triangle given by

\[iu^!\mathbb{Q}_X \rightarrow \mathbb{Q}_X \rightarrow j_*j^*\mathbb{Q}_X,
\]

which gives rise to a Gysin LES

\[H^{*-d}(Z) \rightarrow H^*(X) \rightarrow H^*(U) \xrightarrow{+1} \cdots.
\]

6.2. **Base change.** Suppose we have the cartesian square of a fibre product

\[
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{f} & Y \\
\downarrow^g & & \downarrow^g \\
X & \xrightarrow{f} & Z
\end{array}
\]

Then:
(1) $g^*f_! \simeq \tilde{f}_!\tilde{g}^* : \text{Sh}(X) \to \text{Sh}(Y)$ on the level of abelian categories; there is thus also the derived version $g^*Rf_! \simeq R\tilde{f}_!\tilde{g}^*$.

(2) $g^!Rf_* \simeq R\tilde{f}_*\tilde{g}^!$.

**Example 53.** $Y = \text{pt} \xrightarrow{\pi} X$ gives

$$Rf!_{(\mathcal{F})}x = R\Gamma_c(f^{-1}(x); \mathcal{F}).$$

Note:

- If $f$ is proper, $f_* = f_!$.
- If $f$ is an open embedding, $f^! = f^*$.
- If $f$ is a topological submersion, $f^! \simeq f^*[d]$.

So in these (and other) cases we get interesting base change results.

### 7. Nearby and vanishing cycles.

Want to study the topology of singular varieties, e.g.,

$$X_0 = f^{-1}(0),$$

where $f : X \to \mathbb{C}$ is a proper holomorphic map.

Here $X$ is a (smooth) complex manifold. Think of $X$ as a parametrized collection of varieties (the fibres), all compact since $f$ is proper. Some fibres, such as $X_0$, may be singular – see Figure 23 for an example.

**Figure 23.** Family over $\mathbb{C}$ with singular fibre at 0.

Assume $0 \in \mathbb{C}$ is an (isolated) singular value, i.e. $X_0$ is singular. Then $f|_{f^{-1}(\Delta^*)}$ is a submersion, where $\Delta \subset \mathbb{C}$ is a small disk and $\Delta^* = \Delta - \{0\}$.

**Example 54.** $f(x, y) = y^2 - x^3 - x^2$ has partial derivatives

$$\frac{\partial f}{\partial x} = 3x^2 - 2x, \quad \frac{\partial f}{\partial y} = 2y,$$

so $df$ is onto except for at $(x, y) = (0, 0)$; this is the pinch point.
For $f$ proper, Ehresmann’s theorem implies that $f|_{f^{-1}(\Delta^*)}$ is a locally trivial fibre bundle, i.e. for all $t \in \Delta^*$, there exists a neighbourhood $U$ of $t$ such that there is a diffeomorphism

$$f^{-1}(U) \xrightarrow{\cong} X_t \times U$$

Let’s think about $\mathcal{H}^i := R^i f_* (\mathbb{Q}_X)$. Recall that the stalk

$$\mathcal{H}^i_t = H^i(X_t) \quad \text{(this equality uses proper base change).}$$

$\mathcal{H}^i|_{\Delta^*}$ is a locally constant sheaf, so there is an automorphism

$$T : \mathcal{H}^i_{t_1} \xrightarrow{\cong} \mathcal{H}^i_{t_1},$$

where $t_1 \in \Delta^*$ is a choice of basepoint. $T$ comes from a geometric monodromy (from Ehresmann’s theorem)

$$X_{t_1} \xrightarrow{\cong} X_{t_1}.$$ 

Choosing a path from $t$ to 0 and flowing along it, we also obtain a specialisation map

$$r_t : X_t \rightarrow X_0.$$ 

This induces

$$H^*(X_0) \rightarrow H^*(X_t)$$

via the restriction map

$$\mathcal{H}^i_0 = \mathcal{H}^i(U) \rightarrow \mathcal{H}^i(V) = \mathcal{H}^i_t$$

where $U, V$ are as in Figure 24. 

Write $r \equiv r_t$ and define

$$\psi_f := Rr_* (\mathbb{Q}_{X_t}),$$

which, up to a shift, is the same as

$$\psi_f^i := Rr_* (\omega_{X_t}).$$

So we have a complex of sheaves $\psi_f \in D^+(X_0)$, and

$$H^* (X_0; \psi_f) = H^*(X_t).$$

Similarly,

$$H^* (X_0; \psi_f^i) = H^*_B (X_t).$$

We can also look at stalks. Consider the zoomed in picture around the singular fibre (Figure 25).

---

**Figure 24.** Restriction to an open away from the singular point.
Then
\[(\psi_f)_x = H^*(\text{Mil}_f,x),\]
where the Milnor fibre is
\[\text{Mil}_f,x = B_{x,\epsilon} \cap X_t\]
where \(\epsilon > |t| > 0\) are small enough, and \(B_{x,\epsilon}\) is a ball of radius \(\epsilon\) around \(x\).

**Why?** Unpacking definitions, using that \(r\) is proper, and using base change,
\[\psi_f)_x = H^*(r^{-1}(x));\]
then finally observe that \(r^{-1}(x)\) is the Milnor fibre up to homotopy (compact core of Milnor fibre).

So \(\psi_f\) is a sheaf on \(X_0\) that tells us about the cohomology of nearby fibres.
If \( x \in X_0 \) is nonsingular, \( r^{-1}(x) \) is a single point, so that
\[
(\psi_f)_x \cong H^*(r^{-1}(x)) \cong \mathbb{Q}.
\]
I.e. the Milnor fibre \( \text{Mil}_{f,x} \) is contractible if \( x \) is nonsingular.

The monodromy map gives us a map of sheaves
\[
T : \psi_f \to \psi_f.
\]
If we take
\[
X \cap B_{x,\varepsilon} \to \Delta := f(X \cap B_{x,\varepsilon}),
\]
then over \( \Delta^*_x \) we have a fibre bundle with generic fibre \( \text{Mil}_{f,x} \). We call this the Milnor fibration. See [M] for an original reference.

**Facts:** If \( x \in X_0 \) is an isolated singularity (no matter how terrible),
\[
\text{Mil}_{f,x} \cong \bigvee_{\mu(x)} S^n,
\]
the wedge of \( \mu(x) \) n-spheres where \( n = \dim \mathbb{C}(X_0) \). We call \( \mu(x) \) the Milnor number.

Define \( \phi_f \) as follows: there is a unit map
\[
\mathbb{Q}_{X_0} \to \psi_f = R(r_t)_*(r_t^*\mathbb{Q}_{X_0})
\]
and this corresponds to the specialisation map
\[
H^*(X_0) \xrightarrow{\text{sp}} H^*(X_t).
\]
Define \( \phi_f = \text{cone}(\text{sp}) \). I.e. we have an exact triangle in \( D^+(X_0) \)
\[
\mathbb{Q}_{X_0} \to \psi_f \to \phi_f \xrightarrow{+1} \cdots
\]
So there is a LES in cohomology
\[
H^*(X_0) \to H^*(X_t) \to H^*(X_0; \phi_f),
\]
as well as a local version
\[
H^*(X_0 \cap B\epsilon,x) \to H^*(\text{Mil}_{f,x}) \to H^*(\phi_{f,x}).
\]
Alternatively, we could use \( \psi^! \) to get
\[
\phi_f^! \to \psi_f^! \to \omega_{X_0}
\]
cones shifted by 1 Borel-Moore chains on nearby fibre Borel-Moore chains on singular fibre
so \( (\phi_f^!)_x \) are the cycles in \( \text{Mil}_{f,x} \) which go to 0 under specialisation. We call this the vanishing cycles sheaf.

**Example 55.** If \( X \) has isolated singularities, \( \phi_f^! \) must be a sum of skyscraper sheaves supported at the singular points,
\[
\phi_f^! = \bigoplus_{x \in X_0} \mathbb{Q}_{x}^{\mu(x)}[+n],
\]
where this is implicitly 0 for nonsingular \( x \).

**Example 56.** Want to compute the Milnor fibration for our example \( y^2 - x^3 - x^2 \) at 0 \( \in \mathbb{C}^2 \) (i.e. studying the collapse of the cylinders to a cone near 0). The Hessian is non-degenerate, so the Morse lemma (holomorphic version) tells us we can change coordinates to
\[
f(u, v) = uv : \mathbb{C}^2 \to \mathbb{C}.
\]
We want to compute the monodromy for $X_t \cong \mathbb{C}^\times$ for $t \neq 0$ – we make this identification via the map

$$z \mapsto \left(z, \frac{t}{z}\right).$$

We also have $X_0 = \mathbb{C} \prod_0 \mathbb{C}$. Then flowing along a lift of $\frac{\partial}{\partial \theta}$ downstairs,

$$T_\theta : X_1 \to X_{e^{i\theta}}, \quad T_\theta(u, v) = \left(e^{i\theta}u, e^{i\theta}v\right)$$

is the parallel transport map. In particular,

$$T_{2\pi} : X_1 \to X_1$$

$$T_{2\pi}(u, v) = (-u, -v)$$

is the antipodal map. So the local monodromy is the identity, since on an odd-dimensional sphere the antipodal map is a rotation,

$$T : H^*(X_1^1) \xrightarrow{id} H^*(X_1^1).$$

What about the global monodromy? See Figure 27.
Figure 27. Global monodromy for $X_1$.

8. A PREVIEW OF THE RIEMANN-HILBERT CORRESPONDENCE.

8.1. $C^\infty$ RIEMANN-HILBERT. Let $X$ be a $C^\infty$-manifold of dimension $n$. Consider the sheaf of rings $\mathscr{C}_X^\infty$ on $X$. Then there is a correspondence

$$
\begin{array}{cccc}
\{ \text{$C^\infty_X$-vector bundles on $X$} \} & \leftrightarrow & \{ \text{Locally free sheaves of $C^\infty_X$-modules} \} \\
(V \to X) & \mapsto & \Gamma(-, V) \\
(V, \mathcal{M} \to X) & \iff & \mathcal{M}
\end{array}
$$

We define the bottom map as follows. Given a locally free $C^\infty_X$-module $\mathcal{M}$, the trivializations

$$
\phi_i^\mathcal{M} : \mathcal{M}|_{U_i} \simto (\mathscr{C}_{U_i}^\infty)^r
$$

give rise to transition functions

$$
c_{ij}^\mathcal{M} = \phi_i \circ \phi_j^{-1}|_{U_i \cap U_j} \in C^\infty(U_i \cap U_j, GL_n).
$$

These can then be glued together to give a vector bundle

$$
V^{\mathcal{M}} = \coprod_i U_i \times \mathbb{C}^r
$$

There is another correspondence

$$
\begin{array}{cccc}
\{ \text{Locally constant sheaves of } \mathbb{C}\text{-vector spaces} \} & \leftrightarrow & \{ \text{Vector bundles with flat connection} \} \\
\mathcal{F} & \mapsto & (\mathscr{C}_X^\infty \otimes_{\mathbb{C}_X} \mathcal{F} = \mathcal{M}, d^{\mathcal{M}}(f \otimes s) = df \otimes s) \\
\ker(d^{\mathcal{M}}) & \leftrightarrow & (\mathcal{M}, d^{\mathcal{M}})
\end{array}
$$

Remark

- The data of the $c_{ij}^\mathcal{M}$ determines a class in $H^1(X; C^\infty(\cdot, GL_n))$.
- In the above, $s$ is a local section of $\mathcal{F}$ and $f$ is a local section of $\mathscr{C}_X^\infty$. 

• The data of a vector bundle with flat connection is equivalent to a class in $\tilde{H}^1(X, GL_n)$ (locally constant transition functions).

Given a flat vector bundle $(\mathcal{M}, d^\#)$, define

$$dR(\mathcal{M})^\bullet = \mathcal{M} \stackrel{d^\#}{\to} \mathcal{M} \otimes \mathcal{O}_X \mathcal{A}_X^1 \to \mathcal{M} \otimes \mathcal{O}_X \mathcal{A}_X^2 \to \cdots \to \mathcal{M} \otimes \mathcal{O}_X \mathcal{A}_X^n,$$

where $\mathcal{A}_X^i$ are the smooth $i$-forms on $X$, and we extend $d^\#$ via the Leibniz rule.

**Proposition 8.1.** $0 \to \ker(d^\#) \to dR(\mathcal{M})^\bullet$ gives a quasi-isomorphism.

**Proof.** Can check exactness locally, and then this is immediately implied by the Poincaré lemma. □

**Corollary 8.2.** $H^*(X; \ker(d^\#)) = H^*(\Gamma(dR(\mathcal{M})^\bullet)) = H^*_{dR}(\mathcal{M}, d^\#)$.

**Proof.** Use that the sheaves $\mathcal{M} \otimes \mathcal{O}_X \mathcal{A}_X^m$ are fine (thus soft, thus acyclic). □

### 8.2. Complex geometry.

Now suppose $X$ is a complex manifold, $\dim \mathbb{C} X = n$. Write

- $\mathcal{O}_X$ for the sheaf of holomorphic functions, and
- $\Omega^n_X$ for the sheaf of holomorphic $m$-forms.

We get a correspondence

$$\begin{array}{ccc}
\{ \text{Locally constant sheaves} \} & \leftrightarrow & \{ \text{Holomorphic vector bundles with flat connection} \} \\
\{ \text{of $\mathbb{C}$-vector spaces} \} & & \{ \text{with $d^\#: \mathcal{M} \to \mathcal{M} \otimes \mathcal{O}_X \Omega_X^1$} \} \\
(\text{locally free $\mathbb{C}_X$-modules}) & & (\text{locally free $\mathbb{C}_X$-modules}) \\
& & (\mathcal{M}, d^\#(f \otimes s) = \bar{\partial} f \otimes s) \\
& & (\mathcal{M}, d^\#)
\end{array}$$

**Remark** $\mathcal{O}_X$-modules are not acyclic for $\Gamma(X; -)$.

**Example 57.** If $X \hookrightarrow \mathbb{C}P^N$ is projective, then $X$ is an algebraic variety, and

Coherent $\mathcal{O}_X$-modules $\leftrightarrow$ Coherent $\mathcal{O}_X^{\text{alg}}$-modules

**Example 58.** If $X$ is Stein, $X \hookrightarrow \mathbb{C}^N$, then coherent $\mathcal{O}_X$-modules are acyclic.

**Definition 25.** An $\mathcal{O}_X$-module $\mathcal{M}$ is called coherent if

1. it is finitely generated as an $\mathcal{O}_X$-module, (i.e. for each $x \in X$ there exists a neighbourhood $U \ni x$ and $\mathcal{O}_U^{\text{gr}} \to \mathcal{M}|_U$; and,
2. for each $U \subseteq X$ open and any

$$\phi : (\mathcal{O}_X|_U)^{\oplus r} \to \mathcal{M}|_U,$$

we have that $\ker(\phi)$ is finitely generated.

**Theorem 8.3** (Oka). $\mathcal{O}_X$ is coherent as a module over $\mathcal{O}_X$.

**Exercise 8.1.** Compare to the smooth case: $\mathcal{C}^\infty_X$ is not coherent as a $\mathcal{C}^{\infty}_X$-module.

**Hint:** Consider multiplication by the function

$$f(x) = \begin{cases} 
e^{-\frac{x}{2}}, & x > 0, \\
0, & x \leq 0. 
\end{cases}$$

**Remark** Coherent $\mathcal{O}_X$-modules are the smallest subcategory of sheaves which
(1) contains vector bundles, and
(2) is closed under finite $\oplus$, kernels and cokernels.

Note that $f^*$ does not preserve $\mathcal{O}$-modules.

**Example 59.** Flat connections on $X = \mathbb{C}^\times$ (i.e. locally constant sheaves). Every holomorphic vector bundle on $\mathbb{C}^\times$ is trivial, so

\[
\left\{ \text{holomorphic flat connections on } \mathbb{C}^\times \right\} \rightarrow \left\{ \text{square matrices of holomorphic 1-forms} \right\}
\]

\[
d^A(s) = ds - Asdz \quad \leftrightarrow \quad A(z)
\]

where $d^A$ is a connection on $\mathcal{O}^r_X$. We know that

\[
\{\text{flat connections on } \mathbb{C}^\times \} \simeq \{\text{Reps of } \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}\} \simeq \{\text{vector space with an automorphism}\}.
\]

So how do we determine the monodromy automorphism? Take $A$ to be $1 \times 1$ in what follows – in general the ideas below require us to consider the *path-ordered exponential*.

\[
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{image}
\caption{Choice of path in $\mathbb{C}^\times$ for parallel transport.}
\end{figure}
\]

Given a connection matrix $A$, let $s_0 \in \mathbb{C}^r$ be a section at 1. We want to find $s(t)$ such that $d^A(s) = 0$, i.e.

\[
s'(t) = A(\gamma(t))\gamma'(t)s(t), \quad \text{and } s(0) = s_0.
\]

To solve this, write

\[
\frac{s'}{s} = A(\gamma(t))\gamma'(t),
\]

and observe\(^1\) that the solution is given by

\[
s(t) = \exp \left( \int_0^t A(\gamma(t))\gamma'(t)dt \right) \cdot s_0.
\]

Hence the monodromy automorphism is

\[
s(2\pi) = \exp \left( \oint_\gamma Adz \right) \cdot s_0.
\]

\(^1\)One can motivate this solution by observing that the LHS is a log-derivative, so we expect naively to see

\[
\log \left( \frac{s(t)}{s_0} \right) = \int_0^t A(\gamma(t))\gamma'(t)dt.
\]
Theorem 8.4 (Cauchy). The monodromy matrix is
\[ M(z) = \exp \left( \text{Res}_0(A) \cdot 2\pi i \right). \]

I.e. we only need to look at connections of the form
\[ B(z) = \frac{\text{Res}_0(A(z))}{z}. \]

Exercise 8.2. It follows that every flat connection on \( \mathbb{C}^\times \) is equivalent to one of the form \( d^B, B \) constant, i.e.
\[ d^B = d + \frac{B}{z}. \]

So, find an invertible matrix \( G(z) \) such that
\[ B = ±G^{-1}AG ± G^{-1}dG, \]
where working out the correct signs is part of the exercise.

9. Constructible sheaves.

We now wish to understand a more general class of sheaves.

Definition 26. A partition \( \mathcal{P} \) of a topological space \( X \) is a collection of disjoint, locally closed subsets of \( X \), \( X_i \), such that
\[ \bigcup_i X_i = X. \]

Definition 27. A sheaf \( \mathcal{F} \in \text{Sh}(X; \mathbb{C}) \) is called constructible (w.r.t. \( \mathcal{P} \)) if \( \mathcal{F}|_{X_i} \) is a locally constant sheaf of finite rank.

Example 60. Let \( f : X \to \Delta \) be a proper holomorphic map such that \( f|_{f^{-1}(\Delta^*)} \) is a submersion. Then \( \mathcal{H}^i = R^i f_*(\mathbb{C}_X) \) is a constructible sheaf on \( \Delta = \Delta^* \cup \{0\} \).

Definition 28. A complex \( \mathcal{F} \in D^+\text{Sh}(X; \mathbb{C}) \) is called a constructible complex if \( \mathcal{H}^i(\mathcal{F}) \) is constructible for each \( i \).

Example 61. Let \( f : S^3 \to S^2 \) be the Hopf fibration. \( R^i f_*(\mathbb{C}_{S^3}) \) is a constructible complex of sheaves w.r.t. the trivial partition,
\[ R^i f_*(\mathbb{C}_{S^3}) = \begin{cases} \mathbb{C}, & i = 0, 1, \\ 0, & \text{else}. \end{cases} \]
Recall that
\[ R^if_*(\mathbb{C}_{S^3}) \not\cong \mathbb{C}_{S^2} \oplus \mathbb{C}_{S^2}[-1]. \]
So,
\[ D^+_{c,\emptyset}(S^2) \not\cong D^+(\text{Sh}_{c,\emptyset}(S^2)), \]
where \( D^+_{c,\emptyset} \) means the bounded derived category of constructible complexes with respect to the empty partition, and \( \text{Sh}_{c,\emptyset} \) means local systems.

Example 62. \( X = \Delta = \Delta^* \cup \{0\} \). Let a partition be given by \( \mathcal{P} = \{\Delta^*, \{0\}\} \), and choose nonzero \( t \in \Delta \). What kind of data describes a constructible sheaf?

Let \( \mathcal{F} \in \text{Sh}_{c,\mathcal{P}}(\Delta) \), and let
\[ V_0 = \mathcal{F}_0, \quad V_i = \mathcal{F}_i. \]

We can consider small open sets to determine \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \),
\[ V_0 = \Gamma(U; \mathcal{F}) = \mathcal{F}(U) \]
\[ V_i = \Gamma(V; \mathcal{F}) = \mathcal{F}(V) \]
where \( U \) and \( V \) are as shown in Figure 29.
So we get a quiver

\[
\begin{array}{c}
V_0 \\
\downarrow \alpha \\
V_t \\
\cup \\
\beta \quad \text{(monodromy)}
\end{array}
\]

with the relation

\[ \beta \alpha = \alpha \quad \text{(sections in } V_t \text{ that extend to } V_0 \text{ must have trivial monodromy).} \]

**Fact:** This data (representation of a quiver with given relation) is *equivalent* to the data of a constructible sheaf.

10. Preliminaries on \( \mathcal{D} \)-modules.

Let \( X \) be a complex manifold. An \( \mathcal{O}_X \)-module is a sheaf of modules for the sheaf of rings \( \mathcal{O}_X \).

**Definition 29.** A **\( \mathcal{D} \)-module** on \( X \) is an \( \mathcal{O}_X \) module \( \mathcal{M} \) together with flat connections

\[
d^\mathcal{M} : \mathcal{M} \to \mathcal{M} \otimes \mathcal{O}_X \Omega^1_X
\]

\[
d^\mathcal{M}(f \cdot m) = df \otimes m + f \otimes d^\mathcal{M}(m), \quad f \in \mathcal{O}_X, m \in \mathcal{M}.
\]

Let \( \mathcal{F}_X \) be the sheaf of holomorphic vector fields. For each vector field \( \xi \in \mathcal{F}_X \),

\[
d^\mathcal{M}_\xi : \mathcal{M} \to \mathcal{M},
\]

and flatness means

\[
d^\mathcal{M}_\xi d^\mathcal{M}_\nu - d^\mathcal{M}_\nu d^\mathcal{M}_\xi = d^\mathcal{M}_{[\xi, \nu]} \quad \forall \xi, \nu \in \mathcal{F}_X.
\]

**Example 63.** If \( x_1, \ldots, x_n \) are local coordinates on \( X \), then

\[
\partial_i := \frac{\partial}{\partial x_i} \in \mathcal{F}_X, \quad [\partial_i, \partial_j] = 0,
\]

and we have commuting operators

\[
d^\mathcal{M}_{\partial_i} : \mathcal{M} \to \mathcal{M}.
\]

For now on we write \( \xi(m) \) or \( \xi m \) instead of \( d^\mathcal{M}_\xi(m) \).

Let \( \mathcal{D}_X \) denote the subsheaf of rings of \( \text{End}_{\mathcal{O}_X}(\mathcal{O}_X) \) generated by
• multiplication by $g \in \mathcal{O}_X$;
• $\mathcal{T}_X$ acting by derivations.

$\mathcal{D}_X$ is called the **ring of differential operators**.

In local coordinates, $P \in \mathcal{D}_X$ looks like

$$P = \sum_{\alpha} f_{\alpha}(x) \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}.$$ 

**Fact:** A $\mathcal{D}$-module $\mathcal{M}$ is the same thing as a $\mathcal{D}_X$-module.

**Definition 30.** A **stratification** of $X$ is a partition whose strata are manifolds (+ other conditions not listed here).

**Theorem 10.1.** There is an equivalence

$$\begin{align*}
\left\{ \text{Bounded constructible complexes for some analytic stratification of } X \right\} & \xrightarrow{\text{Riemann-Hilbert correspondence}} D^b \left\{ \text{regular holonomic } \mathcal{D}_X\text{-modules} \right\} \\
\text{dR}(\mathcal{M}) & \xleftarrow{\sim} \mathcal{M}
\end{align*}$$

Sitting inside of these categories we have

$$\begin{align*}
\mathcal{D}_c(X; \mathbb{C}) & \xrightarrow{\sim} D^b(\mathcal{D}_X\text{-mod}^h) \\
\text{Perv}(X; \mathbb{C}) & \xleftarrow{\sim} \mathcal{D}_X\text{-mod}^h
\end{align*}$$

where the bottom left category is the category of perverse sheaves.

We would like to understand this correspondence.

**10.1. Differential equations.** A (linear) differential operator on $X$ is

$$Pu = 0 \quad \text{where} \quad P \in \mathcal{D}_X.$$ 

Given $P \in \mathcal{D}_X$, define

$$\mathcal{M}_P = \mathcal{D}_X/\mathcal{D}_X \cdot P, \quad \text{a left } \mathcal{D}_X\text{-module.}$$

**Warning!** $\mathcal{D}_X$ is non-commutative, so there is a distinction between right and left modules.

$\mathcal{M}_P$ represents the following functor,

$$\mathcal{I} \mapsto \{ u \in \mathcal{I} \mid Pu = 0 \};$$

$\mathcal{I}$ is thought of as a space of functions in which we might look for solutions. Or, more specifically, $\mathcal{I}$ is some space of functions on $X$ which is a $\mathcal{D}_X$-module.

**Example 64.** $\mathcal{O}_X, \mathcal{I}_X$ the space of Schwarz functions, Sobolev spaces, etc.

Then

$$\text{Hom}_{\mathcal{D}_X}(\mathcal{M}_P, \mathcal{I}) = \{ u \in \mathcal{I} \mid Pu = 0 \}.$$ 

**Example 65.** $X = \mathbb{C}$, coordinate $z$.

1. $P = \partial_z$ gives the left $\mathcal{D}$-module

$$\mathcal{M}_P = \mathcal{D}_X/\mathcal{D}_X \cdot \partial_z = \mathcal{O}_X.$$
(2) For \( P = z \), what is the \( \mathcal{D} \)-module \( \mathcal{M}_P = \mathcal{D}_X/\mathcal{D}_X \cdot z \)? Note that the \( \mathcal{D} \)-modules \( \mathcal{M}_P \) are cyclic; i.e. generated as a \( \mathcal{D}_X \)-module by one section \( u \), the image of 1 in the quotient. So we can think of the module as
\[
\mathcal{M}_P = \mathcal{D}_X \cdot u.
\]
For instance, for the prior example, \( \mathcal{M}_\partial = \mathcal{D}_X \cdot 1_X, \partial_z(1_X) = 0 \). What about for this example? We have
\[
\mathcal{M}_z = \mathcal{D}_X \cdot u, \quad z \cdot u = 0.
\]
We call this
\[
u = \delta(z), \quad \text{the } \delta \text{-function.}
\]
For us, \( \delta(z) \) is just the name for the generator of \( \mathcal{M}_z \), which satisfies \( z \cdot \delta(z) = 0 \). You can make sense of this with distributions if you want, but we won’t have to in this course.

Then \( \mathcal{M}_z \) will be generated by
\[
\partial_z \delta(z) = \delta'(z), \ldots, \partial_z^n \delta(z) = \delta^{(n)}(z), \ldots
\]
So
\[
\mathcal{M}_z = \mathbb{C}\langle \delta, \delta', \ldots \rangle \cong \mathbb{C}[\partial_z],
\]
i.e. the \( \mathcal{D} \)-module of constant coefficient differential operators.

(3) \( \mathcal{D}_X \cdot z^\lambda \) with \( \lambda \in \mathbb{C} \). We won’t worry (yet) about troubles with solutions being multivalued. The object \( z^\lambda \) solves a differential equation,
\[
\partial_z(z^\lambda) = \lambda z^{\lambda - 1}
\]
\[
z \partial_z(z^\lambda) = \lambda z^\lambda
\]
\[
\Rightarrow (z \partial_z - \lambda)z^\lambda = 0.
\]
If \( \lambda \notin \mathbb{Z} \), then
\[
\mathcal{D}_X \cdot z^\lambda = \mathcal{D}_X/\mathcal{D}_X(z \partial_z - \lambda).
\]
On \( \mathbb{C}^x \) this gives a flat connection
\[
d^\# = d + \lambda \frac{1}{2} dz
\]
with monodromy \( e^{2\pi i \lambda} \). Why?
\[
\partial_z(f(z)z^\lambda) = \frac{\partial f}{\partial z} z^\lambda + \frac{\lambda}{2} dz(z^\lambda).
\]
One can compute that the stalk at 0 is
\[
(\mathcal{D}_X z^\lambda)_0 = 0.
\]

10.2. The ring \( \mathcal{D}_X \). We now wish to study the ring structure on \( \mathcal{D}_X \); recall from above that this is the sheaf \( \mathcal{D}_X \) on a complex manifold \( X \),
\[
\mathcal{D}_X \subset \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{O}_X).
\]
which is generated by \( \mathcal{O}_X \subset \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{O}_X) \) acting by multiplication, and \( \mathcal{F}_X = \text{Der}(\mathcal{O}_X, \mathcal{O}_X) \subset \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{O}_X) \).

**Note:** If \( \theta \in \mathcal{F}_X, f \in \mathcal{O}_X \), let \( [\theta, f] = \theta f - f \theta \). Then
\[
[\theta, f](g) = \theta(f g) - f \theta(g) = \theta(f) g + f \theta(g) - f \theta(g) = \theta(f) g.
\]
So
\[
[\theta, f] = \theta(f) \in \mathcal{O}_X \subset \mathcal{D}_X.
\]

**Definition 31.** We say \( P \in \mathcal{D}_X \) has **order** \( \leq m \) if \( P \) can be written as a sum of \( \theta_1 \cdots \theta_1, \quad \theta_1 \in \mathcal{F}_X, \quad l \leq m \).

Define
\[
\mathcal{D}_X(m) = \{ \text{differential operators of order } \leq m \}.
\]

**Proposition 10.2.** \([\mathcal{D}_X(m), \mathcal{D}_X(0)] \subset \mathcal{D}_X(m - 1)\). Observe that \( \mathcal{D}_X(0) = \mathcal{O}_X \).
Definition 32 (Alternative definition due to Grothendieck). Given a commutative \( \mathbb{C} \)-algebra \( A \), define a \( \mathbb{C} \)-algebra \( \mathcal{D}_A \subseteq \text{End}_C(A) \) as follows:

- \( \mathcal{D}_A(0) := A \subseteq \text{End}_C(A) \)
- \( \mathcal{D}_A(m) := \{ P \in \text{End}_C(A) \mid [P, f] \in \mathcal{D}_A(m - 1) \forall f \in A = \mathcal{D}_A(0) \} \)

**Remark** This definition makes sense for a sheaf of commutative \( \mathbb{C} \)-algebras \( \mathcal{A} \).

**Example 66.** If \( \mathcal{A} = \mathcal{O}_X \) then \( \mathcal{D}_{\mathcal{O}_X} = \mathcal{D}_X \). This is not hard, but not trivial. E.g. for \( m = 1 \) we suppose \( P : \mathcal{O}_X \rightarrow \mathcal{O}_X \) has the property \([P, f] \in \mathcal{O}_X \). Then we claim that \( P - P(1) \) is a derivation.

**Exercise 10.1.** Prove the claim of the example.

**Example 67.** If \( A \) is the coordinate ring of a smooth affine algebraic variety \( \text{Spec}(A) \), we call \( \mathcal{D}_A \) the ring of algebraic differential operators.

More generally:

- If \( X^{\text{alg}} \) is a smooth variety over \( \mathbb{C} \), then \( \mathcal{D}_X^{\text{alg}} \) is a sheaf of rings in the Zariski topology.
- If \( X^{\text{alg}} \) is smooth, then \( \mathcal{D}_X^{\text{alg}} \) is generated by \( \mathcal{O}_X^{\text{alg}} \) and \( \mathcal{J}_X^{\text{alg}} \).
- **But** if \( X \) is singular this is no longer true in general.

**Example 68.** Let \( A = \mathbb{C}[x_1, \ldots, x_n] \). Then

\[
\mathcal{D}_{\mathbb{C}[x_1, \ldots, x_n]} := W_n, \quad \text{the Weyl algebra.}
\]

This is a ring generated by symbols

\( x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, \)

subject to the relations

\( [x_i, x_j] = 0, \quad [\partial_i, \partial_j] = 0, \quad [\partial_i, x_j] = \delta_{ij}. \)

**Example 69.** Let

\[
A = \mathcal{O}_n := (\mathcal{O}_{\mathbb{C}^n})_0 =: \mathbb{C}\{\{x_1, \ldots, x_n\}\},
\]

(think of as power series with positive radius of convergence). Define \( \mathcal{D}_n := \mathcal{D}_{\mathcal{O}_n} \), and note that at a point \( x \in X \) a complex manifold,

\[
(\mathcal{D}_X)_x \cong \mathcal{D}_n \quad \text{after picking coordinates around} \ x.
\]

**Remark** There is a PBW type theorem for \( \mathcal{D}_n \),

\[
\mathcal{D}_n = \bigoplus_{\alpha} \mathcal{O}_n \cdot \frac{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}{\partial^{\alpha}, \alpha=(\alpha_1, \ldots, \alpha_n)}
\]

where \( \partial_i = \frac{\partial}{\partial x_i} \). I.e. any \( P \in \mathcal{D}_n \) has a unique expression of the form

\[
P = \sum f_\alpha(x) \partial^\alpha.
\]

The order of \( P \) as a differential operator is the weight

\[|\alpha| = \alpha_1 + \cdots + \alpha_n.\]

**Proposition 10.3.**

1. \( \mathcal{D}_n \) is a filtered ring,

\[
\mathcal{D}_n(0) \subseteq \mathcal{D}_n(1) \subseteq \mathcal{D}_n(2) \subseteq \cdots \quad \text{and} \quad \mathcal{D}_n(m)\mathcal{D}_n(k) \subseteq \mathcal{D}_n(m + k).
\]

**Note that this is not graded, as the bracket can reduce the degree, e.g. \([\partial x, x] = 1\).**

2. The associated graded ring of \( \mathcal{D}_n \) is

\[
\text{Gr}(\mathcal{D}_n) := \bigoplus_{m=0}^\infty \mathcal{D}_n(m)/\mathcal{D}_n(m-1).
\]

The symbol maps are the quotients

\[
s_m : \mathcal{D}_n(m) \to \mathcal{D}_n(m)/\mathcal{D}_n(m-1) =: \text{Gr}(\mathcal{D}_n)(m).
\]
This is graded, 
\[ \text{Gr}(\mathcal{D}_n)(m) \cdot \text{Gr}(\mathcal{D}_n)(k) \subseteq \text{Gr}(\mathcal{D}_n)(m+k). \]

Note that \( \sigma_1(x\partial_x) = \sigma_1(\partial_x x) \), for instance. Then we have that 
\[ \text{Gr}(\mathcal{D}_n) = \mathcal{O}_n[\xi_1, \ldots, \xi_n], \]

where \( \xi_i = \sigma_1(\partial_i) \). Observe that this is a commutative ring, which is holomorphic in \( \mathbb{C}^n \) and polynomial in the dual space \((\mathbb{C}^n)^*\).

(3) Define \( \mathcal{D}^{op}_n \) to have the same underlying vector space as \( \mathcal{D}_n \), but with multiplication rule 
\[ P \cdot^{op} Q := QP, \quad P, Q \in \mathcal{D}^{op}_n. \]

Then we have \( \mathcal{D}_n \cong \mathcal{D}^{op}_n \) via the map 
\[ f \in \mathcal{O}_n \xrightarrow{a} f \in \mathcal{O}_n \]

so that 
\[ \sum_{\alpha} f_{\alpha}(x)\partial^\alpha \mapsto \sum_{\alpha} (-1)^{\alpha} \partial^\alpha f_{\alpha}(x). \]

\( \mathcal{D}_X \) is also a filtered sheaf of rings, 
\[ \text{Gr}(\mathcal{D}_X) \xrightarrow{\text{Sym}_{\mathcal{O}_X}(\mathcal{T}_X)} \pi_*\mathcal{O}_{T^*X} \]

where \( \pi : T^*X \rightarrow X \).

To look up (if interested): There is also a ring \( \mathcal{E}_X \) of \textit{micro differential operators}.

\[ \sigma_m(P)\sigma_k(Q) - \sigma_k(Q)\sigma_m(P) = \sigma_{m+k}([P,Q]) = 0. \]

Then the PBW theorem implies that 
\[ \mathcal{D}_n(m)/\mathcal{D}_n(m-1) \cong \bigoplus_{|\alpha|=m} \mathcal{O}_n[\xi_1^{a_1} \cdots \xi_n^{a_n}]. \]

With respect to the third part of the proposition: first recall that left \( \mathcal{A}^{op} \)-modules are the same thing as right \( \mathcal{A} \)-modules. In general (e.g. \( \mathcal{X} \) a complex manifold), we have that \( \mathcal{O}_X \) is a left \( \mathcal{D}_X \)-module. We also have a holomorphic line bundle (locally trivial sheaf of \( \mathcal{O}_X \)-modules of rank 1), \( \Omega^n_X \).

Claim: \( \Omega^n_X \) is naturally a right \( \mathcal{D}_X \)-module.

Motivation: Given \( \eta \in \Omega^n_X \), \( f \in \mathcal{O}_X \), and ignoring for the moment the fact that we are working with holomorphic top forms, consider the desired module structure 
\[ \int (\eta \cdot \theta)f = \int \eta(\theta(f)), \quad \theta \in \mathcal{T}_X. \]

I.e. we want to define \( (\eta \cdot \theta) \) such that 
\[ (\eta \cdot \theta)(f) - \eta(\theta(f)) \]

is exact.

Definition 33. \( \eta \cdot \theta = -\mathcal{L}_\theta(\eta) \), the Lie derivative on top forms.
Then
\[-\mathcal{L}_\theta(\eta) = -d\iota_\theta(\eta) - \iota_\theta(d\eta) = -d\iota_\theta(\eta)\]
since $\eta$ is a top form.

**Claim:** This defines a right $\mathcal{D}$-module structure on $\Omega^n_X$.

**Proposition 10.4.** There is an equivalence of categories
\[
\mathcal{D}_X \text{-mod} \longrightarrow \mathcal{D}_X^{\text{op}} \text{-mod}
\]
\[
\mathcal{M} \longrightarrow \Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{M}
\]

11. **Algebraic geometry.**

Let $X$ be a complex manifold, $\mathcal{O}_X$ the sheaf of holomorphic functions on $X$.

**Definition 34.** A closed subset $Z \subseteq X$ is called analytic if $Z$ is locally of the form
\[V(f_1, \ldots, f_r) = \{x \mid f_1(x) = \cdots = f_r(x) = 0\},\]
where $f_1, \ldots, f_r \in \mathcal{O}_X$.

Given an analytic subset $Z \subseteq X$ we define a sheaf of ideals
\[\mathcal{I}_Z = \{f \in \mathcal{O}_X \mid f|_Z = 0\} \subseteq \mathcal{O}_X.
\]
Note that $\mathcal{I}_Z$ is coherent, since locally $\mathcal{I}_Z = (f_1, \ldots, f_r)$ (by definition).

Note also that $\mathcal{I}_Z$ is radical, i.e. if $f^N \in \mathcal{I}_Z$ then $f \in \mathcal{I}_Z$. I.e.
\[\mathcal{I}_Z = \sqrt{\mathcal{I}_Z} = \{f \mid f^N \in \mathcal{I}_Z \text{ for some } N > 0\}.
\]

**Theorem 11.1** (Analytic Nullstellensatz).
\[
\left\{ \begin{array}{c}
\text{coherent sheaves of} \\
\text{radical ideals}
\end{array} \right\} \sim \left\{ \begin{array}{c}
\text{closed analytic subsets} \\
Z \subseteq X
\end{array} \right\}
\]
\[
\mathcal{I} \longmapsto V(\mathcal{I})
\]
\[
\mathcal{I}_Z \Longleftrightarrow Z
\]

In fact, for any coherent ideal sheaf $\mathcal{I}$,
\[\mathcal{I}_{V(\mathcal{I})} = \sqrt{\mathcal{I}}.
\]

The ring of germs of holomorphic functions at $x \in X$ is $\mathcal{O}_{X,x}$ – it is a noetherian local ring, with maximal ideal the functions vanishing at $x$. Prime ideals in $\mathcal{O}_{X,x}$ correspond to germs of irreducible analytic subsets of $X$ near $x$.

Given an $\mathcal{O}_X$-module $\mathcal{F}$, define
\[\text{supp}^0(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\}.
\]
and let $\text{supp}(\mathcal{F})$ be its closure. There is a sheaf of ideal $\text{Ann}(\mathcal{F}) \subseteq \mathcal{O}_X$,
\[\text{Ann}(\mathcal{F}) := \{f \in \mathcal{O}_X \mid f \cdot s = 0 \quad \forall s \in \mathcal{F}\}.
\]

**Proposition 11.2.** If $\mathcal{F}$ is of finite type (locally finitely generated over $\mathcal{O}_X$), then
\[\text{supp}(\mathcal{F}) = V(\text{Ann}(\mathcal{F})).
\]
In particular, it is an analytic subset.
Proof. Suppose $F_x = 0$, i.e. $x \notin \text{supp}(F)$. Choose local generators $s_1, \ldots, s_r$ near $x$. Note that

$$F_x = 0 \implies (s_1)_x = \cdots (s_r)_x = 0.$$ 

So there exists some $U \ni x$ such that $F|_U = 0$. So,

$$\text{Ann}(F)|_U = \mathcal{O}_U,$$

and so $V(\text{Ann}(F)|_U) = \emptyset$. In particular, $x \notin V(\text{Ann}(F))$. Conversely, if $F_x \neq 0$, take $0 \neq s_x \in F_x, f \in \text{Ann}(F)$. Then $f_x \cdot s_x = 0$ implies $f(x) = 0$ (since if $f(x) \neq 0$ then $f$ is invertible on a neighbourhood of $x$, and $f_x$ would be invertible, implying that $s_x = 0$ – a contradiction). □

Example 70. Let $X = \Delta = \{ |x| < 1 \}$. Let $F = \mathcal{O}_X/(x^k)$. Then $\text{Ann}(F) = (x^k), \sqrt{\text{Ann}(F)} = (x)$, and $\text{supp}(F) = V(x) = \{ 0 \} \subset \Delta$.

Example 71. As above, but let

$$F' = \bigoplus_{i=1}^k \mathcal{O}_X/(x).$$

Both examples are skyscraper sheaves supported at zero with $k$-dimensional stalks, but the are not the same sheaf.

Given an analytic set $i : Z \subseteq X$ we get a sheaf of rings

$$i^{-1}(\mathcal{O}_X/I_Z),$$

where $i^{-1}$ denotes the functor we previously had called $i^*$ (from here on out we adopt this change in notation). $Z$ is called a complex analytic variety, and decomposes as

$$Z = Z^{\text{reg}} \cup Z^{\text{sing}},$$

where the nonsingular points $Z^{\text{reg}}$ form a complex manifold. $Z$ is said to be irreducible if whenever we express $Z$ as a union of analytic subsets

$$Z = Z_1 \cup Z_2,$$

we either have $Z = Z_1$ or $Z = Z_2$.

Fact: $Z = \bigcup_i Z_i$ where the $Z_i$ are irreducible and the union is locally finite.

11.1. Cycle of a coherent sheaf. Suppose $\mathcal{F} \in \text{Coh}(X)$, the category of coherent sheaves, and $Z$ is an irreducible component of $\text{supp}(\mathcal{F})$. We define the multiplicity of $\mathcal{F}$ along $Z$, $m(\mathcal{F}; Z) \in \mathbb{Z}_{>0}$ as follows.
Pick $x \in Z^\text{reg}$ (i.e. a smooth point). Then $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module, and $p = \mathcal{I}_{Z,x}$ is a prime ideal in $\mathcal{O}_{X,x}$ (prime because $x$ is regular in $Z$). Localize at $\mathcal{I}_{Z,x}$. Then $\mathcal{F}_{x,p}$ is an $\mathcal{O}_{X,x,p}$-module (i.e. invert everything in $\mathcal{O}_{X,x} - p$). Recall that $\text{length}_A(M)$ is the length of the longest chain of submodules $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$.

Then we define

$$m(\mathcal{F}; Z) = \text{length}_{\mathcal{O}_{X,x,p}}(\mathcal{F}_{x,p}).$$

(Claim that this is independent of the choice of $x$.)

More explicitly: $\mathcal{F}_x$ has a filtration

$$0 = (\mathcal{F}_x)_0 \subseteq (\mathcal{F}_x)_1 \subseteq \cdots \subseteq (\mathcal{F}_x)_k = \mathcal{F}_x,$$

such that

$$(\mathcal{F}_x)_i/(\mathcal{F}_x)_{i-1} \cong \mathcal{O}_X/p_i$$

for some prime ideals $p_i \subseteq \mathcal{O}_{X,x}$. Then

$$m(\mathcal{F}; Z) = \text{number of times } p \text{ appears as a } p_i.$$

In the algebraic setting, we can take $\xi_Z$ the generic point of $Z$. Then localize at $\xi_Z$, to obtain $\mathcal{F}_{\xi_Z}$ as an $\mathcal{O}_{X,\xi_Z}$-module. Then the multiplicity is the length of this module.

**Definition 35.** The **cycle** of $\mathcal{F}$ is

$$\text{Cyc}(\mathcal{F}) = \sum_{Z \subset \text{supp}(\mathcal{F})} m(\mathcal{F}; Z) \cdot [Z]$$

where the sum is locally finite, and is taken over the irreducible components of $\text{supp}(\mathcal{F})$. We can also define

$$\text{Cyc}_d(\mathcal{F}) = \sum_{\dim(Z) = d} m(\mathcal{F}; Z) \cdot [Z].$$

**Figure 30.** Decomposition of the support into irreducible components.
Finally, we define
\[ m_d(\mathcal{F})_x := \sum_{Z \ni x, \dim(Z) = d} m(\mathcal{F}; Z). \]

**Example 72.** \( X = \Delta, \mathcal{F} = \mathcal{O}_X/(x^k), \mathrm{Ann}(\mathcal{F}) = (x^k). \) So \( \text{supp}(\mathcal{F}) = \{0\}, \) but
\[ \text{Cyc}(\mathcal{F}) = k \cdot [0] = [0] + \cdots + [0], \]
\( \text{i.e.} \ m(\mathcal{F}, \{0\}) = k. \) A chain of length \( k \) is
\[ 0 \to \mathcal{O}_X/(x) \to \mathcal{O}_X/(x^2) \to \cdots \to \mathcal{O}_X/(x^k). \]
We have the same result for \( \mathcal{F}' = \bigoplus_{i=1}^k \mathcal{O}_X/(x), \) i.e.
\[ \text{Cyc}(\mathcal{F}') = k \cdot [0]. \]
So the cycle does **not** distinguish between these two different sheaves.

**12. \( \mathcal{D} \)-modules: Singular support and filtrations.**

Recall that \( \mathcal{D}_X \) is a sheaf of rings on \( X, \) and we have a filtration
\[ F_0 \mathcal{D}_X \subseteq F_1 \mathcal{D}_X \subseteq \cdots; \]
previously we called \( F_i \mathcal{D}_X = \mathcal{D}_X(i), \) the differential operators of order \( \leq i. \) Recall that
\[ \text{Gr}^F(\mathcal{D}_X) = \text{Sym}_* \mathcal{O}_X(\pi^{-1}X) \subseteq (\pi^{-1}X)_* \mathcal{O}_{T^*X}. \]
If \( \mathcal{M} \) is a coherent \( \mathcal{D}_X \)-module, we want to define a subset, the **singular support**, \( SS(\mathcal{M}) \subseteq T^*X, \)
a closed analytic conic subset (i.e. it is stable under the \( \mathbb{C}^* \) action on \( T^*X \) given by \( t \cdot (x, \xi) = (x, t\xi) \)). We will also define the associated cycle of \( SS(\mathcal{M}) \) in \( T^*X, \) which we call \( CC(\mathcal{M}) \) (the **characteristic cycle**).

Assume we have a filtration \( F_i \mathcal{M} (F_k(\mathcal{D}_X) \cdot F_i(\mathcal{M}) \subseteq F_{k+i}(\mathcal{M})) \) such that
\[ \text{Gr}^F(\mathcal{M}) \text{ is coherent as a Gr}^F(\mathcal{D}_X)\text{-module.} \]
Locally we can pick generators \( u_1, \ldots, u_r \) of \( \mathcal{M} \) and define
\[ F_k \mathcal{M} := \bigoplus_{i=1}^r F_k(\mathcal{D}_X) \cdot u_i. \]
\( \text{Gr}(\mathcal{M}) \) is a coherent \( \text{Sym}_\mathcal{D}(\mathcal{I}_X)\text{-module, and we extend scalars} \)
\[ \text{Gr}(\mathcal{M})^\sim := \mathcal{O}_{T^*X} \otimes_{\mathcal{O}_X} \text{Sym}(\mathcal{I}_X) \pi^{-1} \text{Gr}(\mathcal{M}) \in \text{Coh}(T^*X). \]
Then
\[ SS(\mathcal{M}) = \text{supp(Gr}(\mathcal{M})^\sim), \]
\[ CC(\mathcal{M}) = \text{Cyc(Gr}(\mathcal{M})^\sim). \]

**Recall:** A (left) \( \mathcal{D}_X \)-module is equivalent to the data of an \( \mathcal{O}_X \)-module with a flat connection.

Let \( \mathcal{M} \) be a \( \mathcal{D}_X \)-module. Say that \( \mathcal{M} \) is **filtered** when
\[ \cdots \subseteq F_i \mathcal{M} \subseteq F_{i+1} \mathcal{M} \subseteq \cdots \]
such that
\begin{enumerate}
  \item \( F_i \mathcal{M} = 0 \) for \( i << 0, \)
  \item \( \bigcup_i F_i \mathcal{M} = \mathcal{M}, \) and
  \item \( F_i \mathcal{D}_X \cdot F_j \mathcal{M} \subseteq F_{i+j} \mathcal{M}. \)
\end{enumerate}
A filtration is called *good* if additionally
\[ \text{Gr}^F(M) := \bigoplus_i (F_i M / F_{i-1} M) \]
is a coherent Gr\(^F(D_X)\) module.

**Remark** The good condition implies that \( M \) is coherent as a \( D_X \)-module.

**Remark** If \( M \) is coherent (as a \( D_X \)-module) then locally it is generated by some sections, \( s_1, \ldots, s_r \). I.e. there exists \( U \) such that
\[ M|_U = D_U \cdot s_1 + \cdots + D_U \cdot s_r. \]

Define
\[ F_i M|_U := F_i D_U \cdot s_1 + \cdots + F_i D_U \cdot s_r. \]

**Claim:** \((M|_U, F)\) is a good filtration. I.e. good filtrations exist locally for a coherent \( D_X \)-module.

Given \((M, F)\) a good filtered \( D_X \)-module,
\[ \text{Gr}^F(M) \text{ is a coherent } \text{Sym}^{\bullet}(O_X)(T_X) = \text{Gr}(D_X)\)-module.

**Aside:** If \( V \) is a finite dimensional vector space over \( \mathbb{C} \), the ring of polynomial functions on \( V \) is
\[ \text{Sym}^{\bullet}(V^*) = \mathbb{C} \oplus V^* \oplus \text{Sym}^2(V^*) \oplus \cdots \]
I.e. if \( e_1, \ldots, e_r \) is a basis for \( V \) with dual basis \( \epsilon_1, \ldots, \epsilon_r \),
\[ \text{Sym}^{\bullet}(V^*) = \mathbb{C}[e_1, \ldots, e_r]. \]

If \( M \) is a Sym\(^\bullet\)(V\(^\ast\))-module, then for \( \text{Ann}(M) \subseteq \text{Sym}^\bullet(V^*) \)
\[ V(\text{Ann}(M)) = \text{supp}(M) \subseteq V. \]

\( V \) is also a \( \mathbb{C} \)-manifold, so we have \( \mathcal{O}_V \). We can think of Sym\(^\bullet\)(V\(^\ast\)) as a constant sheaf of rings. Then we have
\[ \mathcal{O}_V \otimes \text{Sym}^\bullet(V^*) = \mathcal{O}_V, \]
which is a coherent sheaf. \( \text{supp}(M) = V(\text{Ann}(M)) \), i.e. we have the same notion of support.

If \( M = \bigoplus_i M_i \) is a graded module over Sym\(^\bullet\)(V\(^\ast\)) = \( \bigoplus_i \text{Sym}^i(V^*) \) (a graded ring), then \( \text{Ann}(M) \) is a homogeneous ideal. Then \( \text{supp}(M) \) is *conic*, i.e. it is preserved by the \( \mathbb{C}^\times \) action on \( V \). *End aside.*

Now, to make sense of \( \text{Sym}^\bullet_{\mathcal{O}_X}(\mathcal{T}_X) \), think of
\[ V = T^*_x X, \quad V^* = T_x X \text{ for each point } x \in X. \]
So if \((M, F)\) is a good coherent \( D_X \)-module, we have that
\[ \text{Gr}^F(M) \text{ is a coherent } \text{Sym}^\bullet_{\mathcal{O}_X}(\mathcal{T}_X)\)-module,
and so we get
\[ \text{Gr}(M)^\sim = \mathcal{O}_{T^* X} \otimes_{\pi_X^{-1}\text{Sym}(\mathcal{T}_X)} \pi_X^{-1}(\text{Gr}^F(M)), \]
which is a coherent \( \mathcal{O}_{T^* X} \)-module, where \( \pi_X : T^* X \to X \) is projection.

**Definition 36.** The *singular support* is
\[ SS(M, F) = \text{supp}(\text{Gr}^F(M)^\sim) = V(\text{Ann}(\text{Gr}^F(M)^\sim)) \subseteq T^* X. \]

**Proposition 12.1.** \( SS(M, F) \) is independent of the choice of good filtration \( F \).

We omit the proof.

**Corollary 12.2.** If \( M \) is a coherent \( D_X \)-module, \( SS(M) \subseteq T^* X \) is well-defined.
Definition 37. The characteristic ideal of $\mathcal{M}$ is

$$J_{\mathcal{M}} := \sqrt{\text{Ann}(\text{Gr}(\mathcal{M})^\sim)} \subseteq \text{Sym}^\bullet(\mathcal{T}_X).$$

The characteristic cycle is

$$CC(\mathcal{M}) = \text{Cyc}(\text{Gr}(\mathcal{M})^\sim).$$

Example 73. Let $\mathcal{M} = \mathcal{D}_X$ and $F$ be the order filtration.

$$\text{Gr}^F(\mathcal{D}_X) = \text{Sym}^\bullet(\mathcal{T}_X)$$

and

$$\text{Gr}^F(\mathcal{D}_X)^\sim = \mathcal{O}_{T^*X},$$

so we have

$$SS(\mathcal{D}_X) = \supp(\mathcal{O}_{T^*X}) = T^*X.$$

Remark Think of $\mathcal{D}_X$ as a noncommutative deformation of $\text{Sym}^\bullet(\mathcal{T}_X) \subseteq \mathcal{O}_{T^*X}$, i.e. that $\mathcal{D}_X$ is a quantization of $T^*X$.

Think of $\mathcal{D}_X$-modules as “living on $T^*X$”. The first approximation to this idea is

$$\mathcal{M} \mapsto SS(\mathcal{M}) \subseteq T^*X.$$

Example 74. Given $P \in \mathcal{D}_X$, let $\mathcal{M}_P = \mathcal{D}_X/\mathcal{D}_X \cdot P$. (I.e. this roughly corresponds to solving the equation $Pu = 0$.) So $P$ could be, for example, $x^2\partial^2_x + \cdots$. Then the principal symbol of $P$ is

$$\sigma(P) = \text{image of } P \text{ in } \text{Gr}_m(\mathcal{D}_X) \text{ where } m \text{ is maximal}.$$ 

Implicitly we are considering the $\partial_\ast$ as coordinates on the cotangent bundle. For instance, $P = x^2\partial^2_x + x\partial_x$ has

$$\sigma(P) = x^2\xi^2,$$

where $(x,\xi)$ are coordinates on $T^*X$ (here $\xi = \sigma(\partial_x)$).

Proposition 12.3. $SS(\mathcal{M}) = V(\sigma(P)) \subseteq T^*X$.

Proof. We have a SES

$$0 \rightarrow \mathcal{D}_X \xrightarrow{P} \mathcal{D}_X \xrightarrow{q} \mathcal{M}_P \rightarrow 0.$$ 

$q$ induces a good filtration on $\mathcal{M}_P$ by taking the image of the filtration on $\mathcal{D}_X$. Then

$$\text{Gr}(\mathcal{M}_P) = \text{Sym}(\mathcal{T}_X)/\langle \sigma(P) \rangle.$$ 

Example 75. Let $X = \Delta = \{|x| < 1\} \subseteq \mathbb{C}$. Take $P = \partial_x$, so $\mathcal{M}_P = \mathcal{O}_X$. Then

$$SS(\mathcal{O}_X) = V(\xi) = X \subset T^*X,$$

where $X$ includes into $T^*X$ as the zero section.

Example 76. Now take $P = x$. Then

$$SS(\mathcal{M}_P) = V(x) = T^*_0X \subseteq T^*X.$$
Figure 31. Singular supports of $\mathcal{O}_X$ and $\mathcal{M}_x$ in $T^*X$.

Example 77. Now take $P = x\partial_x$. Then $\sigma(P) = x\xi$, so $SS(\mathcal{M}_P) = V(x\xi) = X \cup T^*_0 X$.

Figure 32. Singular support of the Euler operator $x\partial_x$.

In the above examples, the characteristic cycles are

$$CC(\mathcal{O}_X) = [X], \quad CC(\mathcal{M}_x) = [T^*_0 X], \quad CC(\mathcal{M}_{x\partial_x}) = [X] + [T^*_0 X].$$

Suppose $(\mathcal{V}, \Delta)$ is a flat connection, i.e. $\mathcal{V}$ is a locally free $\mathcal{O}_X$-module of finite rank.

**Proposition 12.4.** $SS(\mathcal{V}) = X \subset T^*X$.

**Proof.** The induced filtration is

$$0 = F_{-1} \mathcal{V} \subset F_0 \mathcal{V} = \mathcal{V} \subset \mathcal{V} \subset \mathcal{V} \subset \cdots,$$

so $Gr^F(\mathcal{V}) = \mathcal{V}$. Locally we can choose a horizontal frame and express $\mathcal{V} = \mathcal{O}_X^{\oplus r}$ as a $\text{Sym}(\mathcal{T}_X)$-module, where $\mathcal{T}_X$ acts by 0. So, $\text{Ann}(\mathcal{V}) = (\mathcal{F}_X)$, and

$$SS(\mathcal{V}) = X, \quad CC(\mathcal{V}) = r \cdot [X].$$

$\square$

In fact, the following are equivalent for a coherent $\mathcal{D}_X$-module $\mathcal{M}$:

1. $\mathcal{M}$ is a flat connection (i.e. locally free $\mathcal{O}_X$-module).
2. $SS(\mathcal{M})$ is contained in the zero section of $T^*X$. 

(3) \( \mathcal{M} \) is coherent as an \( \mathcal{O}_X \)-module.

**Lemma 12.5** (Bernstein’s Lemma). If \( \mathcal{M} \) is a coherent \( \mathcal{D}_X \)-module,

\[
\dim(SS(\mathcal{M})) \geq \frac{1}{2} \dim(T^*X).
\]

We defer the proof to later.

Recall that if \((W^{2n}, \omega)\) is a symplectic vector space, \(V \subseteq W\) a subspace, then \(V\) is called

- **isotropic** if \( \omega|_V = 0 \) (which implies that \( \dim(V) \leq n \)),
- **coisotropic** if \( \omega|_{V^\perp} = 0 \) (which implies that \( \dim(V) \geq n \)),
- **Lagrangian** if it is isotropic and coisotropic (in which case \( \dim(V) = n \)).

**Theorem 12.6** (Gabber’s Theorem). \( SS(\mathcal{M}) \) is coisotropic in \( T^*X \).

**Definition 38.** A \( \mathcal{D}_X \)-module is called **holonomic** if \( \dim(SS(\mathcal{M})) = \dim(X) \).

By Gabber’s theorem, this is equivalent to \( SS(\mathcal{M}) \) being Lagrangian in \( T^*X \).

### 13. Holonomic \( \mathcal{D} \)-modules.

We now consider a generalization of the theory of differential equations.

If \( Pu = 0 \), \( P \in \mathcal{D}_C \), we have that the space of solutions \( u \in \mathcal{O}_C \) is finite dimensional. If \( X \) is a complex manifold, recall that a (coherent) \( \mathcal{D}_X \)-module \( \mathcal{M} \) is called **holonomic** if \( SS(\mathcal{M}) \subseteq T^*X \) is Lagrangian (equivalently, \( \dim SS(\mathcal{M}) = \dim(X) \)).

**Example 78.** If \( \dim X = 1 \) and \( \mathcal{D}_X = \mathcal{D}_X / \mathcal{D}_X \cdot P = \mathcal{M}_P \) for \( P \in \mathcal{D}_X \), then

\[
SS(\mathcal{M}_P) = V(\sigma(P)) \subseteq T^*X,
\]

and \( \dim SS(\mathcal{M}_P) = 1 \) unless \( P \) is constant.

**Theorem 13.1** (Kashiwara, PhD thesis). Let \( \mathcal{M} \) be a holonomic \( \mathcal{D}_X \)-module. Then we have two complexes of sheaves,

\[
Sol(\mathcal{M}) := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X),
\]

“solutions of \( \mathcal{M} \)”,

\[
DR(\mathcal{M}) := \mathcal{M} \xrightarrow{\nabla} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_X \rightarrow \cdots \rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^n_X,
\]

“de Rham complex”.

These are both constructible complexes with regards to some “nice” analytic stratification of \( S \). In particular, the stalks of both complexes are finite dimensional.
14. Functors for $\mathcal{D}$-modules.

Recall that for sheaves we had the “6-functor formalism”: $*$- and $!$- adjoint pushforwards and pullbacks, $\mathcal{E}xt$ and $\otimes$. We would like similar adjoint functors for $\mathcal{D}$-modules – we will see, however, that these may not always exist, and even when they do exist they will only be adjoint for holonomic $\mathcal{D}$-modules.

Given a holomorphic map of complex manifolds, $f : X \to Y$, we will define functors between the bounded derived categories of $\mathcal{D}$-modules

$$
\begin{array}{ccc}
D^b(D_Y) & \xrightarrow{f_*} & D^b(D_X) \\
\downarrow{f^!} & & \downarrow{f_*} \\
D^b(D_X) & \xleftarrow{f^*} & D^b(D_Y)
\end{array}
$$

**Beware:** In general these are **not** adjoint!

We recall and compare previous situations. Some notation has been changed from previous lectures to avoid overloading.

14.0.1. $\mathbb{C}_X$-modules.

$$
\begin{array}{ccc}
\text{Sh}(X; \mathbb{C}) & \xrightarrow{f_*} & \text{Sh}(Y; \mathbb{C}) \\
\downarrow{f^!} & & \downarrow{f_*} \\
\text{Sh}(Y; \mathbb{C}) & \xleftarrow{f^*} & \text{Sh}(X; \mathbb{C})
\end{array}
$$

$f^*$ is exact, and is left adjoint to $f_*$ (which is therefore left exact). These give rise to derived functors $f^{-1}$ and $Rf_*$.

14.0.2. $\mathcal{O}$-modules.

$$
\begin{array}{ccc}
\mathcal{O}_X\text{-mod} & \xrightarrow{f_*} & \mathcal{O}_Y\text{-mod} \\
\downarrow{f^\varnothing} & & \downarrow{f_*} \\
\mathcal{O}_Y\text{-mod} & \xleftarrow{f^!} & \mathcal{O}_X\text{-mod}
\end{array}
$$
where
\[ f_\ast^\#(\mathcal{F}) := \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{F}). \]

\( f_\ast^\# \) is left adjoint to \( f_\bullet \), so is right exact. \( \mathcal{O}_X \) is generally not \( f^{-1}(\mathcal{O}_Y) \)-flat, so \( f_\ast^\# \) is not exact. Thus, we have a left derived functor \( Lf_\ast^\# \). Similarly, \( f_\bullet \) is left exact, so gives a right derived functor \( Rf_\bullet \).

14.0.3. Some examples.

1. If \( V \) is a vector bundle on \( Y \),
\[ f^{-1}(V) = V \times_Y X \longrightarrow V \]
\[ X \longrightarrow Y \]
and
\[ \Gamma(-; f^{-1}(V)) = f_\ast^\#(\Gamma(-; V)). \]

2. If \( i : \{x\} \hookrightarrow X \), \( \mathcal{F} \in \mathcal{O}_X\text{-mod} \), then
\[ i^{-1}(\mathcal{F}) = \mathcal{F}_x \text{, the stalk of } \mathcal{F} \text{ at } x. \]
This is often an \( \infty \)-dimensional space of convergent power series. Conversely and comparatively,
\[ i^\#_\ast(\mathcal{F}) = \left( \frac{\mathcal{O}_{X,x}}{m_x} \right) \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x = \mathcal{F}_x / \mathcal{F}_x \cdot m_x, \]
which we call the fibre of \( \mathcal{F} \) at \( x \). If \( \mathcal{F} \) is a vector bundle, this is literally the fibre.

14.0.4. \( \mathcal{D} \)-modules. Define a functor
\[ f^\circ : \mathcal{D}_Y\text{-mod} \to \mathcal{D}_X\text{-mod} \]
as follows. Let
\[ f^\circ(\mathcal{M}) = f_\ast^\#(\mathcal{M}) \]
as an \( \mathcal{O}_X \) module. Then to define a \( \mathcal{D}_X \)-module structure, we need to say how \( v \in T_X \) acts on \( m \in f_\ast^\#(\mathcal{M}). \)
Tangent vectors pushforward, \( T_{X,x} \to T_{Y,f(x)} \), and these can be put together to give a map
\[ \mathcal{F}_x \to f_\ast^\#(\mathcal{F}_Y) \]
\[ v \mapsto \tilde{v} \]
\[ f_\ast^\#(\mathcal{M}) = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{M}). \] Given \( g \otimes m, g \in \mathcal{O}_X \) and \( m \in f^{-1}(\mathcal{M}), \)
\[ v \cdot (g \otimes m) = v(g) \otimes m + g \otimes \tilde{v}(m). \]
Now define (since \( f^\circ \) is right exact) a left derived functor
\[ f^\dagger := Lf^\circ. \]

14.1. Transfer bimodule. Define
\[ \mathcal{D}_{X \to Y} := f^\circ(\mathcal{D}_Y). \]
I.e.
\[ \mathcal{D}_{X \to Y} = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{D}_Y), \]
so this carries a left \( \mathcal{D}_X \)-module action, and a right \( f^{-1}(\mathcal{D}_Y) \)-module action. Then we have functors (of left modules)
\[ \mathcal{D}_Y\text{-mod} \xrightarrow{f^\circ} f^{-1}(\mathcal{D}_Y)\text{-mod} \xrightarrow{\mathcal{D}_X \otimes_{f^{-1}(\mathcal{D}_Y)} (-)} \mathcal{D}_X\text{-mod}. \]

We call \( \mathcal{D}_{X \to Y} \) the transfer bimodule. Then
\[ f^\dagger = \mathcal{D}_{X \to Y} \otimes_{f^{-1}(\mathcal{D}_Y)} (f^{-1}(-)), \]
the left derived tensor product. We also have a functor in the opposite direction, but for right modules:
\[ \mathcal{D}_X^{\text{op}}\text{-mod} \xrightarrow{(\cdot) \otimes_{\mathcal{O}_X} \mathcal{D}_{X \to Y}} f^{-1}(\mathcal{D}_Y)^{\text{op}}\text{-mod} \xrightarrow{f^\bullet} \mathcal{D}_Y^{\text{op}}\text{-mod}. \]
Recall that there is a nontrivial equivalence of categories between $D_X\text{-mod}$ and $D_X^{\text{op}}\text{-mod}$ (and that this is a special property of $D$-modules, not something that holds for arbitrary noncommutative rings). To get a functor on left $D$-modules we take

$$D_X \otimes - \text{mod} \xrightarrow{\sim} D_X \otimes_{\mathcal{O}_X} (\mathcal{O}_X^{-1}) \otimes D_Y \text{-mod} \xrightarrow{f_*} D_Y \text{-mod}$$

Define

$$f_*^{\text{dR}} : D^b(D_X) \rightarrow D^b(D_Y)$$

$$f_*^{\text{dR}}(\mathcal{M}) := Rf_*(\mathcal{O}_{D_X^{-1}} \otimes \mathcal{O}_{D_Y} \mathcal{M})$$

What is $D_{X \leftarrow Y}$? There is an equivalence of categories

$$D_X - f^{-1}(D_Y)\text{-bimodules} \simeq D_X^{\text{op}} - f^{-1}(D_Y)^{\text{op}}\text{-bimodules} = f^{-1}(D_Y) - D_X\text{-bimodules}.$$ 

Then $D_{X \rightarrow Y} \leftrightarrow D_{Y \leftarrow X}$ under this equivalence.

14.2. Closed embeddings. Let $i : Z \hookrightarrow X$ be a closed embedding (recall that this implies $i_*$ is exact). Consider

$$i^\circ : D_X\text{-mod} \rightarrow D_Z\text{-mod}$$

$$i^\circ(\mathcal{M}) = D_{Z \rightarrow X} \otimes_{i^{-1}}(D_X) i^{-1}(\mathcal{M}),$$

and

$$i^! : D_X\text{-mod} \rightarrow D_Z\text{-mod}$$

$$i^!(\mathcal{M}) = i^{-1} \text{Hom}_{D_X}(D_{X \leftarrow Z}, \mathcal{M})$$

Lemma 14.1 (Kashiwara’s Lemma). There is a pair of adjoint functors $(i_\circ, i^!)$

$$D_Z\text{-mod} \xleftrightarrow{i_\circ, i^!} D_X\text{-mod}$$

which induce an equivalence of categories between

$$D_Z\text{-mod} \xleftrightarrow{i_\circ, i^!} (D_X\text{-mod})_Z$$

where the right hand side is the category of $D_X$-modules supported on $Z$.

Remark This is not true for $\mathcal{O}$-modules.

Example 79. Consider the inclusion of a point $\{x\} \hookrightarrow X = \mathbb{C}$. Then $\mathcal{O}_{\{x\}}\text{-mod} = \text{Vect}$, while $(\mathcal{O}_X\text{-mod})_{\{x\}}$ contains nontrivial extensions, such as

$$0 \rightarrow \mathcal{O}_{X,x}/m_x \rightarrow \mathcal{O}_{X,x}/m_x^2 \rightarrow \mathcal{O}_{X,x}/m_x \rightarrow 0.$$ 

So $(\mathcal{O}_X\text{-mod})_{\{x\}}$ is not semisimple, thus not equivalent to $\text{Vect}$.

Lemma 14.2 (Kashiwara’s Lemma part II). $(i_\circ, i^!)$ also gives an equivalence on the level of derived categories.
14.3. **Where do the transfer bimodules come from?** $\mathcal{D}_X$-modules “live on $T^*X$”, as a deformed version of quasicoherent sheaves. Given a map $f : X \to Y$, we don’t get a map between cotangent bundles. Instead we get a correspondence or span

$$T^*X \xleftarrow{\rho_f} T^*Y \times_Y X \xrightarrow{\pi_f} T^*Y.$$ 

In fact $T^*X$ and $T^*Y$ are symplectic, and thinking of the above as a map into the product $T^*X \times T^*Y$, the middle object is in fact Lagrangian. We then have

$$\begin{align*}
T^*X &\xleftarrow{\rho_f} T^*Y \times_Y X \xrightarrow{\pi_f} T^*Y \\
\mathcal{D}_X &\xrightarrow{\text{quantization}} \mathcal{D}_{X \to Y} \text{ or } \mathcal{D}_{Y \to X} \xrightarrow{\text{quantization}} \mathcal{D}_Y
\end{align*}$$

We can understand the reverse map, e.g. $\mathcal{D}_X \to T^*X$; $\mathcal{D}_X$ has a filtration as a $\mathcal{D}_X$-module, and the associated graded is functions on $T^*X$.

14.3.1. **Case 1**: $f = i : X \hookrightarrow Y$ a closed embedding.

**Example 80.** See Figure 34.

In more generality, $X$ is defined by a sheaf of ideals $\mathcal{I}_X \subseteq \mathcal{O}_Y$. Then

$$\begin{align*}
\mathcal{O}_X &= i^{-1}(\mathcal{O}_Y / \mathcal{I}_X) \\
\mathcal{D}_{X \to Y} &= \mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\mathcal{D}_Y = i^{-1}(\mathcal{D}_Y / \mathcal{I}_X \cdot \mathcal{D}_Y)
\end{align*}$$

**Exercise 14.1.** Show that $\mathcal{D}_{Y \hookrightarrow X} = i^{-1}(\mathcal{D}_Y / \mathcal{D}_Y \cdot \mathcal{I}_X)$.

**Definition 39.** We define two underived functors:

$$\begin{align*}
i^\circ(\mathcal{N}) &= \mathcal{D}_{X \to Y} \otimes_{i^{-1}\mathcal{D}_Y} i^{-1}\mathcal{N} = i^{-1}(\mathcal{N} / \mathcal{I}_X \cdot \mathcal{N}), \\
i^\bullet(\mathcal{N}) &= \mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y \hookrightarrow X}, i^{-1}\mathcal{N}) \in \mathcal{D}_X\text{-mod}
\end{align*}$$

**Remark** Here and following, $\mathcal{N} \in \mathcal{D}_Y\text{-mod}$ and $\mathcal{M} \in \mathcal{D}_X\text{-mod}$.

Observe that

$$i_\circ(\mathcal{M}) = i_\bullet(\mathcal{D}_{Y \hookrightarrow X} \otimes_{\mathcal{D}_X} \mathcal{M});$$

there is no need to derive, since pushforward along a closed embedding is exact.
Remark

- $i^\circ$ uses the forward facing bimodule $\mathcal{D}_{X \to Y}$.
- $i^\dagger$ and $i_\circ$ use the backwards facing bimodule $\mathcal{D}_{Y \leftarrow X}$.

**Proposition 14.3.**

1. $i_\circ$ is exact ($\mathcal{D}_{Y \leftarrow X}$ is flat over $\mathcal{D}_X$), $i^\circ$ is right exact, and $i^\dagger$ is left exact.
2. $i^\dagger = Li_\circ = Ri^\circ[d]$ where $d = \text{codim}(X \hookrightarrow Y)$.
3. $i^\dagger$ is right adjoint to $i_\circ$, and moreover $Ri^\circ$ is right adjoint to $Li_\circ = i^\dagger R$.
4. $i_\circ$ induces an equivalence of categories $\mathcal{D}_X\text{-mod} \overset{i_\circ}{\longrightarrow} (\mathcal{D}_Y\text{-mod})_X$

Again, there is a derived statement:

$$\mathcal{D}_X\text{-mod} \overset{i_\circ}{\longrightarrow} (\mathcal{D}_Y\text{-mod})_X$$

**Example 81.** Let's look at a special case:

$X = \{z = 0\} \subseteq Y$ a hypersurface.

We have local coordinates $y_1, \ldots, y_n = z$ on $Y$ (so the last coordinate cuts out the hypersurface $X$). In this case the transfer bimodule is $\mathcal{D}_{X \to Y} = i^{-1}(\mathcal{D}_Y / z \cdot \mathcal{D}_Y)$.

This is resolved by

$$i^{-1}(\mathcal{D}_Y / z \cdot \mathcal{D}_Y) \simeq (i^{-1}\mathcal{D}_Y \overset{\cdot z}{\longrightarrow} i^{-1}\mathcal{D}_Y)$$

and similarly

$$\mathcal{D}_{Y \leftarrow X} \simeq (i^{-1}\mathcal{D}_Y \overset{\cdot z}{\longrightarrow} i^{-1}\mathcal{D}_Y).$$

So we have an explicit description

$$i^\circ(N) = \mathcal{H}om_{i^{-1}\mathcal{D}_Y} (i^{-1}\mathcal{D}_Y / i^{-1}\mathcal{D}_Y \cdot z), N) = \ker(z : N \to N).$$

Now, as a right $\mathcal{D}_X$-module,

$$\mathcal{D}_{Y \leftarrow X} = \mathbb{C}[\partial_z] \otimes_{\mathbb{C}} \mathcal{D}_X.$$

The difference between this and $\mathcal{D}_Y$ is that we don’t have functions of the coordinate $z$ (which vanishes on $X$) – however, we still have vector fields in the $\partial_z$-direction. Note that $\mathcal{D}_{Y \leftarrow X}$ is free as a right $\mathcal{D}_X$-module.

Thus, the functor $i_\circ$ is given by

$$i_\circ(M) = i_\bullet(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X} M) = \mathbb{C}[\partial_z] \otimes_{\mathbb{C}} M.$$

Think: $i_\circ$ is “fattening $M$ up” along the line $\langle \partial_z \rangle$.

What is the $\mathcal{D}_Y$-module structure on this?

$$i_\circ M = \cdots \oplus \partial_z^2 M \oplus \partial_z M \oplus M \overset{\cdot z}{\longrightarrow} 0.$$

Claim that since $z \cdot M = 0$, the action of the other $\cdot z$ is determined by commutations with the $\partial_z$. Define $E := z\partial_z$ the Euler operator.

Claim that $\partial_z^2 M$ is the $(-i - 1)^{\text{th}}$-eigenspace of $E$.

$$E = z\partial_z = \partial_z z - 1,$$

and $z \cdot M = 0$. So $E$ acts on $M$ as multiplication by $-1$. Then we can check that

- multiplication by $\partial_z$ lowers the eigenvalue by 1, and
- multiplication by $z$ increases the eigenvalue by 1.
Claim: $i^\natural i_o \mathcal{M} \cong \mathcal{M}$.

Proof. The $z$ and $\partial_z$ operations are isomorphisms on $i_o \mathcal{M}$, except for in the last place where $z\mathcal{M} = 0$. So

$$i^\natural i_o \mathcal{M} = \ker(z : i_o \mathcal{M} \to i_o \mathcal{M}) = \mathcal{M}.$$ 

$\square$

Remark $i^\natural i_o \mathcal{M} \leftarrow \mathcal{M}$ will be the unit for an adjunction.

Now suppose $\mathcal{N} \in D_Y\text{-mod}$. Look at

$$\mathcal{N}^i := \{i^{th} \text{ eigenspace of } E \text{ acting on } \mathcal{N}\}.$$ 

- We have no guarantee that $i$ is integral.
- We have no idea whether this is bounded at either end:

$$\cdots \oplus \mathcal{N}^{i-2} \oplus \mathcal{N}^{i-1} \oplus \mathcal{N}^i \oplus \mathcal{N}^{i+1} \oplus \mathcal{N}^{i+2} \cdots$$

Note:

- This might not be all of $\mathcal{N}$ (e.g. $\mathcal{N}^{i+\frac{1}{2}}$ could exist).
- $\ker(z) \subseteq \mathcal{N}^{-1}$, so we see that the counit for the adjunction is

$$\cdots \oplus \partial_z \ker(z) \oplus \ker(z) = i_o i^\natural \mathcal{N} \to \mathcal{N}.$$ 

Now assume $\text{supp}(\mathcal{N}) \subseteq X$. I.e.

$$z^N \cdot m = 0 \text{ for all } m \in \mathcal{N}, \; N >> 0.$$ 

If $z \cdot m = 0$ then $m \in \mathcal{N}^{-1}$. By an induction argument, if $z^N \cdot m = 0$ then

$$m \in \mathcal{N}^{-N} \oplus \cdots \oplus \mathcal{N}^{-1}.$$ 

Exercise 14.2. In this case, $i_o i^\natural \mathcal{N} \xrightarrow{\sim} \mathcal{N}$.

Hence in this case we have shown an equivalence

$$\{\mathcal{D}\text{-modules on } X\} \leftrightarrow \{\mathcal{D}\text{-modules on } Y \text{ supported on } X\}$$

$$\mathcal{D}_{X\text{-mod}} \xrightarrow{i^\natural} \xleftarrow{i_o} (D_{Y\text{-mod}})_X$$

Remark If $X \hookrightarrow Y$ is a closed embedding but not a hypersurface, it is cut out by a collection of functions (locally coordinate functions), $X = \{z_1 = \cdots = z_r = 0\}$. Let

$$K(z_i) = (i^{-1}(\mathcal{O}_Y \xrightarrow{z_i} i^{-1}(\mathcal{O}_Y))).$$

Then we define the Koszul complex of $X \subseteq Y$ to be

$$K(z_1) \otimes \cdots \otimes K(z_r).$$
14.4. **A concrete calculation.** Let’s compute

\[ p^\text{dR}_!: D^b(D_X) \to D^b(C), \]

where \( p : X \to \text{pt} \). We have the transfer bimodules

\[ D_X \to \text{pt} = \mathcal{O}_X \text{ left } D \text{-module, right module for constant sheaf) } \]

and

\[ \mathcal{O}_X \leftarrow D_X = \mathcal{O}_X \otimes \Omega^{top}_X = \Omega^{top}_X = \Omega^n_X, \]

where \( n = \text{dim}(X) \). So

\[ p^\text{dR}_!(\mathcal{M}) = \Omega^n_X \otimes L D_X^\text{op} \mathcal{M}. \]

How do we compute \( \otimes^L \)? We either need to replace \( \Omega^n_X \) of \( \mathcal{M} \). To do this uniformly for all \( \mathcal{M} \) we will replace \( \Omega^n_X \) by a complex of locally free \( D_X \text{-modules} \) (right \( D_X \text{-modules})

We do this by taking a left module resolution of \( \mathcal{O}_X \), then we will use the dualizing sheaf to produce the desired resolution.

**Lemma 14.4.** There are (locally free) resolutions

\[ D_X \otimes \mathcal{O}_X \Lambda^n \mathcal{I}_X \to \cdots \to D_X \otimes \mathcal{O}_X \Lambda^2 \mathcal{I}_X \to D_X \to \mathcal{O}_X = D_X / D_X \mathcal{I}_X \]

\[ P \otimes v \]  

\[ P \otimes (v_1 \wedge v_2) \to P v_1 \otimes v_2 - P v_2 \otimes v_1 \]

\[ -P \otimes [v_1, v_2] \]

and

\[ D_X = \Omega^0_X \otimes \mathcal{O}_X \to \cdots \to \Omega^{n-1}_X \otimes \mathcal{O}_X \to \Omega^n_X \otimes \mathcal{O}_X \to D_X \text{DR}(D_X) \to \Omega^n_X \]

the de Rham complex of \( D_X \) through of as a \( D_X \text{-module}. \)

The proof of these claims reduces to the commutative case. I.e. equip everything with compatible good filtrations. Then take associated graded – you will see this is

\[ i_* \mathcal{O}_X \in \mathcal{O}_{T^*_X} \text{-mod,} \]

where \( i : X \to T^*_X \) is the zero section.

Why is this enough? There is an algebraic lemma to prove: if you have a good filtered complex of \( D \text{-modules} \), it is exact iff its associated graded is exact. This is a common technique – reduce a non-commutative problem to a problem in the associated graded.

So, we now have a quasi-isomorphism (so equality in the derived category)

\[ \Omega^L_X \otimes_{D_X} \mathcal{M} \simeq (\Omega^\bullet_X \otimes \mathcal{O}_X D_X) \otimes_{D_X} \mathcal{M} = \Omega^\bullet_X \otimes \mathcal{O}_X \mathcal{M} = \text{DR}(\mathcal{M}). \]

**Caution:** We have changed out grading convention. Now, \( \text{DR}(\mathcal{M}) \) is concentrated in degrees \(-n, -n+1, \ldots, 0\).

Now, to compute \( p^\text{dR}_!(\mathcal{M}) \) we take the pushforward, i.e.

\[ p^\text{dR}_!(\mathcal{M}) = R \Gamma(\text{DR}(\mathcal{M})). \]

**Think:** Analogue/generalization of taking the de Rham cohomology.

**Example 82.** Let \( \mathcal{M} = \mathcal{O}_X \). Then

\[ \text{DR}(\mathcal{M}) = \mathcal{O}_X \to \Omega^1_X \to \Omega^2_X \to \cdots \to \Omega^n_X. \]

Without the grading shift, this is a resolution of the constant sheaf, i.e.

\[ \text{DR}(\mathcal{O}_X) \simeq \mathbb{C}[n]. \]
Then
\[ p^\text{dR}_*(\mathcal{O}_X) = R\Gamma(X; \mathcal{C}_X[n]) = H^{*-n}(X; \mathcal{C}) \]
in degrees $-n$ to $n$.

**Remark** This will make Poincaré duality look symmetric around 0.

**Remark** Could play the same game with $\mathcal{O}_X$ replaced by a local system.

More generally, suppose $f : X \to Y$ is smooth (i.e. is a submersion – this is smoothness in the algebraic geometric sense). We define $\Omega^k_{X/Y}$ as

\[ 0 \to f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0. \]

Locally (by the implicit function theorem) $X \cong Y \times Z$, so $\Omega^1_{X/Y}$ has elements of the form $f(y, z)dz$. Then we can form

\[ \Omega^k_{X/Y} = \bigwedge^k \mathcal{O}_X \Omega^1_{X/Y}, \]

which carries a de Rham differential $d_{X/Y}$ (apparent from the local form of the elements).

**Proposition 14.5.**

\[ f^\text{dR}_*(\mathcal{M}) = Rf_* (DR_{X/Y}(\mathcal{M})). \]

**Remark** For any $f : X \to Y$ we can factor as

\[ X \xrightarrow{\Gamma_f} X \times Y \xrightarrow{p_2} Y, \]

a closed embedding (via the graph of $f$) followed by a submersion.

**Remark** We say a submersion is smooth because the fibres have the structure of a smooth manifold – describing a map by the properties of its fibres is a general principle in algebraic geometry.

Let $f : X \to Y$ be smooth. Then we have

\[ T^*Y \times_Y X \xrightarrow{\pi_f} T^*X \xleftarrow{\rho_f} T^*Y \]

**Proposition 14.6.**

1. $f^\circ : \mathcal{D}_Y\text{-mod} \to \mathcal{D}_X\text{-mod}$ is exact (so $f^\dagger \simeq f^\circ$).
2. $f^\circ$ preserves coherent $\mathcal{D}$-modules.
3. $\text{SS}(f^\circ \mathcal{M}) = \rho_f \pi_f^{-1} (\text{SS}(\mathcal{M}))$.
4. $f^\dagger[-d]$ is left adjoint to $f^\text{dR}_*$, where $d$ is the relative dimension of $f$.

**Example 83.** Consider $X \to \text{pt}$. The singular support must lie in the zero section, as shown in Figure 35.
In this case, observe that $\rho_f$ is a closed embedding and $\varpi_f$ is smooth.

Proof of part 1. When $f$ is smooth it is in particular flat, so $\mathcal{O}_X$ is flat as an $f^{-1}\mathcal{O}_Y$-module. Thus $f^*$ is exact. \hfill \Box

We defer the rest of the proof to later.

15. (Verdier) Duality for $\mathcal{D}$-modules.

Let $\mathcal{M} \in \mathcal{D}_X$-mod. We want to define a dual to $\mathcal{M}$, so our first obvious guess is

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X).$$

This is a right $\mathcal{D}_X$-module; i.e. a $\mathcal{D}_X^{op}$-module. To get a left $\mathcal{D}_X$-module, tensor with the inverse of the dualizing sheaf,

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} (\Omega_X^n)^\vee \in \mathcal{D}_X$$

Define

$$\mathcal{D}_X(\mathcal{M}) := (R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} (\Omega_X^n)^\vee)[\dim X].$$

Example 84. Let $X = \mathbb{C}$, $\mathcal{M}_p = \mathcal{D}_X / \mathcal{D}_X \cdot P$, $P \in \mathcal{D}_X$. Then there is a quasi-isomorphism

$$\mathcal{M}_p \simeq \mathcal{D}_X \xrightarrow{P} \mathcal{D}_X.$$

So

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_p, \mathcal{D}_X) = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X / \mathcal{D}_X \cdot P, \mathcal{D}_X)$$

$$\mathcal{D}_X \xrightarrow{P} \mathcal{D}_X$$

Definition 40. For $\mathcal{M} \in \mathcal{D}_X$-mod and $\mathcal{N} \in \mathcal{D}_X^{op}$-mod, define

$$\mathcal{M}^\ast = \mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X^n \in \mathcal{D}_X^{op}$$

$$\mathcal{M}^{\dagger} = (\Omega_X^n)^\vee \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathcal{D}_X$$
Since $\Omega^n_X \cong \mathcal{O}_X$ for $X = \mathbb{C}$, we therefore have
\[
\mathcal{D}_X(\mathcal{M}) = \mathcal{D}_X / \mathcal{D}_X \cdot P^*,
\]
in degree 0 due to the degree shift in the definition. Here $P^*$ is the adjoint of $P$ – recall that there is a nontrivial isomorphism
\[
\begin{align*}
\mathcal{D}_X & \xrightarrow{\sim} \mathcal{D}_X^{\text{op}} \\ x & \longmapsto x \\ \partial_x & \longmapsto -\partial_x \\ x\partial_x & \longmapsto -\partial_x \cdot x
\end{align*}
\]
and $P^*$ is the image of $P$ under this map.

**Proposition 15.1.** Let $D^{b}_{\text{coh}}(\mathcal{D}_X)$ be the derived category of complexes of $\mathcal{D}_X$-modules with coherent cohomology. Then

- $\mathcal{D}_X : D^{b}_{\text{coh}}(\mathcal{D}_X) \xrightarrow{\sim} D^{b}_{\text{coh}}(\mathcal{D}_X)^{\text{op}}$.
- $\mathcal{D}_X \circ \mathcal{D}_X \simeq \text{id}_X$.
- $\mathcal{M}$ is holonomic (in degree 0) if and only if $\mathcal{D}_X(\mathcal{M})$ is in degree 0.

15.1. **Big Picture.** We started the semester with sheaves on a space $X$:

$$
\begin{array}{ccc}
H^*(X) & \xrightarrow{f^*} & H^*_c(X) \\
\downarrow & & \downarrow \\
X \text{ a topological space} & \xrightarrow{H_*} & H_*^\text{BM}(X)
\end{array}
$$

$D^b(\mathcal{C}_X)$, the derived category of sheaves on $X$.

The 6-functor formalism gave us a nice framework by which to perform calculations in topology: recall that given $f : X \to Y$ we had functors
\[
Rf_*, \quad Rf_!, \quad f^*, \quad f^!,
\]
where

- $Rf_*$ is right adjoint to $f^*$, and
- $Rf_!$ is left adjoint to $f^!$.

There is also a duality functor, $\mathbb{D}_X$. Its cohomology sheaves $\mathcal{H}^i \mathbb{D}_X(\mathcal{F})$ are given by the sheafification of
\[
U \mapsto H^i_c(U; \mathcal{F}^\vee).
\]

We constructed $\mathbb{D}_X(\mathcal{C}_X) = \omega_X$. Then
\[
\mathbb{D}_X(\mathcal{F}) = R\mathcal{H}om_{\mathcal{C}_X}(\mathcal{F}, \omega_X).
\]

It is useful to use this duality functor to talk about relations between our functors. To do this we have to restrict the category we look at:

$D^b_c(\mathcal{C}_X)$ = bounded derived category of constructible complexes (with respect to some stratification).

Then $\mathbb{D}_X$ gives an equivalence
\[
\mathbb{D}_X : D^b_c(\mathcal{C}_X) \xrightarrow{\sim} D^b_c(\mathcal{C}_X)^{\text{op}}, \quad \mathbb{D}_X \circ \mathbb{D}_X \simeq \text{id}.
This is a quite nontrivial self-duality.

**Example 85.** $D_X(C_X) = \omega_X$, and $\text{Hom}(\mathcal{F}, \mathcal{G}) = \text{Hom}(D\mathcal{G}, D\mathcal{F})$.

**Fact/slogan:** Commuting past the duality functor turns $!$ into $\ast$,

- $Rf_* \circ D_X \simeq D_Y \circ Rf_!$.
- $f^* \circ D_Y \simeq D_X \circ f^!$.

This will, for instance, swap various kinds of (co)homology around. So this is a “sheafy” version of Poincaré duality.

**Definition 41.** $D_X$ is called the Verdier duality functor.

We would like a similarly nice (6 functor) formalism for $\mathcal{D}$-modules. As motivations, recall the picture of the Riemann-Hilbert correspondence (Figure 36).

**Example 86.** Given $f : \mathbb{C}^N \to \mathbb{C}$ a holomorphic function, consider $f^{-1}(0) = Z \subseteq \mathbb{C}^N$. Supposing some niceness properties (e.g. $0$ an isolated singular value), we had some nice properties of the following functors and sheaves. For instance, consider

$$Rf_*(\mathcal{C}_{\mathbb{C}^N}) \in D_c^b(\mathbb{C})$$

We considered the fibre $Z$ as the central fibre $X_0$ in a family as per Figure 37.

---

**Figure 37.** Studying a space $Z$ as the central fibre in a family.
We then studied $Rf_*(\mathcal{O}_X)$, but also some more interesting sheaves of vanishing and nearby cycles, $\phi_f, \psi_f \in D^b_c(\mathbb{C}_X^0)$.

To compute $f!, f^*, Rf_!, Rf^*, \psi_f, \phi_f$, we will ask what they correspond to under the Riemann-Hilbert correspondence. We will also ask what the singular support $\text{SS}(\mathcal{M})$ of a $\mathcal{D}$-module corresponds to under Riemann-Hilbert.

**Plan:**

- Functoriality for coherent and holonomic $\mathcal{D}$-modules.
- Regular singularities, Riemann-Hilbert.
- perverse sheaves, intersection cohomology.
- $V$-filtration, nearby and vanishing cycles, specialization to the normal cone.

15.1.1. *Aside: Algebraic versus analytic $\mathcal{D}$-modules.* We’ve been talking about $X$ a complex manifold, with sheaves $\mathcal{O}_X$ of holomorphic functions and $\mathcal{D}_X$ of holomorphic differential operators.

We could instead have started with $X$ a smooth algebraic variety (over $\mathbb{C}$). Then talking complex points gives a complex manifold, but this does not have a converse – e.g. the disk $\Delta = \{x | |x| < 1\}$ is not an algebraic variety.

<table>
<thead>
<tr>
<th>$X$ a complex manifold (complex topology)</th>
<th>$X$ a smooth algebraic variety over $\mathbb{C}$ (Zariski topology)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{O}_X^{\text{alg}}$ holomorphic functions</td>
<td>$\mathcal{O}_X^{\text{alg}}$ algebraic functions</td>
</tr>
<tr>
<td>$\mathcal{D}_X^{\text{alp}}$ holomorphic differential operators</td>
<td>$\mathcal{D}_X^{\text{alp}}$ algebraic differential operators</td>
</tr>
</tbody>
</table>

*Table 1. Algebraic versus analytic comparison.*

**Example 87.** $X = \mathbb{C}^n = \mathbb{A}^n \supset U_f = \mathbb{A}^n - \{f = 0\}$. Then\(^2\)

$$\mathcal{O}_X^{\text{alg}} = \mathbb{C}[x_1, \ldots, x_n]$$

$$\mathcal{O}_X^{\text{alg}}(U_f) = \mathbb{C}[x_1, \ldots, x_n][f^{-1}]$$

This is manifestly a much smaller space than $\mathcal{O}_X^{\text{alp}}$, the holomorphic functions on $\mathbb{C}^n$.

Now, $\mathcal{D}_X^{\text{alp}}$ contains elements of the form

$$\sum_i f_i(x) \partial^i$$

where the $f_i$ are holomorphic functions – i.e. elements which are a power series in $x$ and a polynomial in $\partial$.

The description of $\mathcal{D}_X^{\text{alp}}$ is much simpler. For example, we can present it in terms of generators and relations:

$$\mathcal{D}_X^{\text{alp}} = \mathbb{C}(x_1, \ldots, x_n, \partial_1, \ldots, \partial_n)/ \sim, \quad [\partial_i, x_j] = \delta_{ij}, \quad [x_i, x_j] = 0 = [\partial_i, \partial_j]$$

15.2. **Base change for $\mathcal{D}$-module functors.** Given $f : X \to Y$, we produced

$$\xymatrix{ D^b(\mathcal{D}_X) \ar[r]^{f^!_{\text{dR}}} & \ar[l]_{f^*_{\text{dR}}} D^b(\mathcal{D}_Y) }$$

where $f^!_{\text{dR}} := f^! [\dim X - \dim Y]$. **Warning:** Recall that these are not necessarily adjoint.

Consider the base change

$$\begin{array}{ccc}
X' &=& X \times_Y Y' \\
\downarrow f &=& \downarrow f \\
Y' &\to& Y
\end{array}$$

\(^2\)Abusing notation by conflating a sheaf with its global sections – okay since we are working with affine varieties.
Theorem 15.2 (Base Change Theorem). \( f^*_\text{dR}g^* \cong \tilde{g}^*_\text{dR}f^*_\text{dR} \).

There are two special cases:

- **f** is proper (e.g. **f** is a closed embedding).
  - \( f^*_\text{dR} \) will preserve coherent complexes.
  - \( f^*_\text{dR} \) is left adjoint to \( f^\dagger \).
  - \( f^*_\text{dR} \circ D_X \cong D_Y \circ f^*_\text{dR} \).

- **f** is smooth (i.e. a submersion) of relative dimension \( d \).
  - \( f^\dagger \) will preserve coherent complexes.
  - \( f^\dagger \) is left adjoint to \( f^*_\text{dR} \).
  - \( f^\dagger \circ D_Y \cong D_X \circ f^\dagger [-2d] \).
  - We have \( f^*_\text{dR} = f^\dagger [-d] \) and \( f^\dagger = f^\dagger [d] \).
  - \( f^\dagger \) is exact.

16. Interpretation of singular support.

**Question:** What does \( \text{SS}(\mathcal{M}) \) mean?

**Answer (to be explained):** It measures directions in \( X \) for which solutions to \( \mathcal{M} \) propagate.

Let \( f: X \to Y \), \( X, Y \) complex manifolds (working in the analytic setting). We have

\[
\begin{array}{ccc}
T^*Y \times_Y X & \xrightarrow{\varpi_f} & T^*Y \\
\rho_f \downarrow & & \downarrow \\
T^*X & \to & T^*X
\end{array}
\]

\( SS(\mathcal{M}) \) lives here

Fix \( \mathcal{M} \in (\mathcal{D}_Y\text{-mod})_{\text{coh}} \).

**Definition 42.** \( f \) is called noncharacteristic for \( \mathcal{M} \) if

\[
\varpi_f^{-1}(SS(\mathcal{M})) \cap T^*_X Y = \emptyset.
\]

Here \( T^*_X Y = \ker(\rho_f) \) (a vector bundle on \( X \), and

\[
T^*_X Y = T^*_X Y - \{0\text{-section}\}.
\]

If \( X \hookrightarrow Y \) is a closed embedding, \( T^*_X Y \) is called the conormal bundle of \( X \hookrightarrow Y \). Another way of understanding this in the closed embedding case is as:

\[
\varpi_f^{-1}(SS(\mathcal{M})) \cap T^*_X Y \subseteq 0\text{-section}.
\]

**Example 88.** If \( f \) is smooth (submersion) then

\[
T^*_X Y = \{0\} \times X.
\]

Thus \( f \) is noncharacteristic for any \( \mathcal{M} \).

**Example 89.** If \( SS(\mathcal{M}) = T^*_Y Y = 0\text{-section} \), any \( f \) is noncharacteristic for \( \mathcal{M} \).

**Example 90.** Consider

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{s} & & \\
Z
\end{array}
\]
where \( f \) and \( g \) are both closed embeddings. Let \( \mathcal{M} = f_{\text{DR}}^!(\mathcal{O}_X) \). When is \( Z \) a noncharacteristic submanifold for \( \mathcal{M} \) (i.e. when is \( g \) noncharacteristic for \( \mathcal{M} \))?

Recall from a previous lecture that
\[ SS(\mathcal{M}) = T^*_XY. \]
This is noncharacteristic if and only if
\[ \varpi^{-1}_g(T^*_XY) \cap (T^*_ZY) = 0 \text{-section}, \]
if and only if
\[ (T^*_XY)_x \cap (T^*_ZY)_x = 0 \quad \text{for all } x \in Z \cap X. \]
I.e. \( SS(\mathcal{M}) \) is noncharacteristic if and only if \( X \triangleleft Z \), as in Figure 38.

![Figure 38. Transverse embedded submanifolds with their conormal bundles.](image)

**Theorem 16.1.** Suppose \( f : X \to Y \) is noncharacteristic for \( \mathcal{M} \).

1. \( f^!(\mathcal{M}) = \mathcal{H}^0 f^!(\mathcal{M}) = f^!(\mathcal{M}) \) (concentrated in degree zero – i.e. not being concentrated in degree zero is a non-transversality condition).
2. \( f^!(\mathcal{M}) \) is a coherent \( D_X \)-module.
3. \( SS(f^!(\mathcal{M})) = p_f(\varpi^{-1}_f(SS(\mathcal{M}))). \)
4. \( f^!(D_Y \mathcal{M}) \cong D_X(f^!(\mathcal{M})). \)

Recall:
- \( f^!(\mathcal{M}) = \mathcal{O}_X \otimes f^{-1}\mathcal{O}_Y f^{-1}\mathcal{M}. \)
- \( f^!_{\text{DR}}(\mathcal{M}) = f^!(\mathcal{M})[\dim(X) - \dim(Y)]. \)

In general, \( f^! \) (or \( f^!_{\text{DR}} \)) does not preserve coherence.

**Example 91.** \( i : \{0\} \hookrightarrow \mathbb{A}^1 \) induces \( i^!(D_{\mathbb{A}^1}) \), which is infinite dimensional.

**Upshot:** When is it reasonable to restrict coherent \( D \)-modules as above? When the \( D \)-module is noncharacteristic.

**Remark** The above theorem still makes sense in the algebraic setting – it might even be easier to prove there.

**Theorem 16.2 (Cauchy-Kovalevskaya-Kashiwara).** Let \( f : X \to Y \) be noncharacteristic for \( \mathcal{M} \), then
\[ \text{Sol}_X(f^!(\mathcal{M})) = f^{-1}\text{Sol}_Y(\mathcal{M}). \]

Recall that
\[ \text{Sol}_Y(\mathcal{M}) = R\mathcal{H}om_{D_X}(\mathcal{M}, \mathcal{O}_Y). \]
Motivated by: \( \mathcal{M} = \mathcal{D}_X / \mathcal{D}_X \cdot P \), where we interpret
\[
\text{Sol}_X(\mathcal{M}) = \{ u \mid Pu = 0 \}.
\]
Consider the special case (the general proof always reduces to this special case):
\[
X \subseteq Y \text{ a hypersurface},
\[
\mathcal{M} = \mathcal{D}_Y / \mathcal{D}_Y \cdot P, \quad P \in \mathcal{D}_Y.
\]
Everything is local, so choose coordinates \( y_1, \ldots, y_n \) on \( X \) such that
\[
X = \{ y_1 = 0 \} \subseteq Y.
\]
What does it mean for \( X \) to be noncharacteristic for \( \mathcal{M} \) (this should be some condition on the differential operator \( P \))? Recall that
\[
\text{SS}(\mathcal{M}) = V(\sigma_m(P)), \quad \text{where } P \text{ is of order } m.
\]
Let \( \xi = \sigma(\partial) \) (a function on the cotangent bundle), and write
\[
P = \sum_{|\alpha| \leq m} a_\alpha(y)\partial^\alpha
\]
\[
\sigma_m(P) \sum_{|\alpha| = m} a_\alpha(y)\xi^\alpha.
\]
\( X \) is noncharacteristic for \( \mathcal{M} \) (or for \( P \)) iff for all \( \xi \in T^*_X Y \sigma_m(P)(\xi) \neq 0 \), iff
\[
\sigma_m(P) = (0, y_2, \ldots, y_n, 1, 0, \ldots, 0) \neq 0.
\]
(Here the first coordinate is 0 since we are on \( X \), and consequently the only conormal direction to \( X \) is in the \( \xi^1 \) position.) In coordinates we write \( P \) as
\[
P = \sum a_{\alpha_1, \ldots, \alpha_n}(y_1, \ldots, y_n)\partial_{y_1}^{\alpha_1} \cdots \partial_{y_n}^{\alpha_n}.
\]
So the above condition means
\[
a_{m,0,\ldots,0}(\alpha, y_2, \ldots, y_n) \neq 0.
\]
Looking in a neighbourhood of \( X \), then, we can invert this term so that, without loss of generality,
\[
P = \partial_1^m + (\text{lower order terms in } \partial_1).
\]
So: \( X \) being noncharacteristic for \( P \) of order \( m \) means that \( P \) is \( m^{th} \) order in the \( X \) direction, plus (potentially) other terms in other directions.

So, consider the claims of the theorem. We have
\[
f^1(\mathcal{M}) = \mathcal{M} /_{-1} y_1 \rightarrow \mathcal{M}.
\]
This has potentially two cohomology groups. We want to show that \( y_1 \cdot \) is injective. This would imply that
\[
f^1(\mathcal{M}) = f^\circ(\mathcal{M}) = \mathcal{M} / y_1 \mathcal{M} = \mathcal{D}_Y / (\mathcal{D}_Y \cdot P + y_1 \cdot \mathcal{D}_Y).
\]
Claim:
\[
\mathcal{D}_Y / (\mathcal{D}_Y \cdot P + y_1 \cdot \mathcal{D}_Y) \cong \mathcal{D}_{X}^{\otimes m}.
\]
Example 92. Consider
\[
f^\circ(\mathcal{M}) = \mathcal{D}_X[\partial_1] / \mathcal{D}_X[\partial_1] \cdot P.
\]
Let \( P = \partial_1^m \). Then we have that
\[
f^\circ(\mathcal{M}) \cong \mathcal{D}_{X}^{\otimes m}.
\]
In general,
\[
1, \partial_1, \partial_1^2, \ldots, \partial_1^{m-1} \quad \text{is a basis for } f^\circ(\mathcal{M}) \text{ as a } \mathcal{D}_X\text{-module}.
\]
Let’s pretend/assume that this corresponds to a proof of the first theorem we wrote down. What about the Cauchy-Kovalevskaya-Kashiwara theorem? The claim there was that

\[
\begin{align*}
    f^{-1}\text{Sol}_Y(\mathcal{M}) & \cong \text{Sol}_X(f \uparrow \mathcal{M}) \\
    f^{-1}R\mathcal{H}om_{D_Y}(D_Y/D_Y \cdot P, \mathcal{O}_Y) & \cong R\mathcal{H}om_{D_X}(D_X^{\oplus m}, \mathcal{O}_X) \\
    f^{-1}(\mathcal{O}_Y_0 \xrightarrow{P(-)} \mathcal{O}_Y) & \cong \mathcal{O}_X^{\oplus m}
\end{align*}
\]

The above map is then

\[
\begin{align*}
    f^{-1}(\mathcal{O}_Y_0 \xrightarrow{P(-)} \mathcal{O}_Y) & \rightarrow \mathcal{O}_X^{\oplus m} \\
    u \in \mathcal{O}_X & \rightarrow (u|_X, \partial_1 u|_X, \ldots, \partial_{m-1} u|_X)
\end{align*}
\]

This isomorphism is essentially an existence and uniqueness statement for an \(m\)th order boundary value problem:

**Theorem 16.3** (Cauchy-Kovalevskaya). The Cauchy problem

\[
\begin{align*}
    \begin{cases}
        Pu &= 0 \\
        u|_X &= v_1 \\
        \partial_1 u|_X &= v_2 \\
        \vdots \\
        \partial_{m-1} u|_X &= v_n
    \end{cases}
\end{align*}
\]

has a unique solution \(u \in f^{-1}\mathcal{O}_Y\).

(Here \(P = \partial_{m} + \cdots\) is noncharacteristic.)

To answer the question, “What does \(SS(\mathcal{M})\) mean?”, we will need a variant of the Cauchy-Kovalevskaya theorem. Let \(\phi : X \rightarrow \mathbb{R}\) be a \(C^\infty\)-function, \(X\) a complex manifold. Suppose

\[X_0 = \phi^{-1}(0)\]

is a smooth real hypersurface.

Let \(\mathcal{M}\) be a coherent \(D_X\)-module such that

\[
SS(\mathcal{M}) \cap \hat{T}_{X_0}^*(X_\mathbb{R}) = \emptyset.
\]

**Remark** This uses the identification

\[
\begin{align*}
    (T^*X_\mathbb{R}) & \cong T^*(X_\mathbb{R}) \\
\end{align*}
\]

\[
\begin{align*}
    (\partial\phi)|_x & \leftrightarrow (d\phi)|_x.
\end{align*}
\]

Then

\[
(R\Gamma_{X_{\geq 0}}(\text{Sol}_X(\mathcal{M})))_{X_0} \simeq 0.
\]

We can picture this as in Figure 39.
Recall that if $k : W \hookrightarrow X$ is a locally closed embedding,

$$R\Gamma_W = Rk_!k^!,$$

$$(-)_W = Rk^!k_!.$$ We have a decomposition of our space into an open and a closed, so letting

$$\mathcal{F}^\bullet := \text{Sol}_X(\mathcal{M}),$$

we have an exact triangle

$$R\Gamma_{X\geq 0}(\mathcal{F}^\bullet) \to \mathcal{F}^\bullet \to R\Gamma_{X<0}(\mathcal{F}^\bullet) \xrightarrow{+1}.$$ Looking stalkwise we see that applying $(-)_{X\leq 0}$ and using the theorem gives

$$(R\Gamma_{X\leq 0}(\mathcal{F}^\bullet))_{X\leq 0} = 0,$$

so,

$$(\mathcal{F}^\bullet)_{\leq 0} \xrightarrow{\sim} R\Gamma_{X\geq 0}(\mathcal{F}^\bullet).$$

We interpret this as saying that (holomorphic) solutions to $\mathcal{M}$ on $X_{\leq 0}$ extend (to holomorphic solutions) over $X_0$.

**Meaning:** Solution to $\mathcal{M}$ on $X_{\leq 0}$ can propagate over a “small’ neighbourhood of $X_0$. The proof of this involves reducing to the case

$$\mathcal{M} = D_X/D_X \cdot P,$$

then expressing the result as a Cauchy problem.


**Definition 43.** Let $\mathcal{F} \in D^b(C_X)$, $X$ a $C^\infty$-manifold. Define the microsupport of $\mathcal{F}$, $\mu s(\mathcal{F})$ as follows. $\mu s(\mathcal{F}) \subseteq T^*X$, and

$$T^*X \ni (x_0, \xi_0) \notin \mu s(\mathcal{F}) \quad \text{if and only if} \quad T^*X \ni (x_0, \xi_0) \notin \mu s(\mathcal{F})$$

There exists an open set $U \subseteq T^*X$ with $(x_0, \xi_0) \in U$ such that for all $x \in X$ and $\phi : X \to \mathbb{R}$ a smooth function with $\phi(x) = 0$, $(d\phi)_x \in U$, we have that $R\Gamma_{\phi \geq 0}(\mathcal{F}) \simeq 0$.

**Theorem 16.4.** If $\mathcal{M}$ is a coherent $D_X$-module, then

$$SS(\mathcal{M}) = \mu s(\text{Sol}_X(\mathcal{M})).$$
The generalized Cauchy-Kovalevskaya theorem provides \( SS(\mathcal{M}) \supseteq \mu_s(\text{Sol}_X(\mathcal{M})) \). The other direction is harder, and we omit it.

Recall that
\[
\text{Sol}_X(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).
\]

**Example 93.** Let \( \mathcal{M} = \mathcal{O}_X \) and recall that there exists a resolution of \( \mathcal{O}_X \) as a \( \mathcal{D}_X \)-module, \( \mathcal{D}_X \otimes_{\mathcal{O}_X} (\Omega^\bullet_X) \). Then
\[
\text{Sol}_X(\mathcal{O}_X) \cong \Omega^\bullet_X \cong \mathcal{C}_X, \quad \text{(by the Poincaré lemma), so,}
\]
\[
SS(\mathcal{O}_X) = X \subseteq T^*X.
\]

**Example 94.** Let \( \mathcal{M} = \mathcal{D}_X \). Then \( \text{Sol}_X(\mathcal{D}_X) = \mathcal{O}_X \) – remember this is not as a \( \mathcal{D} \)-module, it is as the sheaf of solutions to a \( \mathcal{D} \)-module. Recall that
\[
SS(\mathcal{D}_X) = T^*X.
\]
Then \( \mu_s(\mathcal{O}_X) = T^*X \). I.e. there are no directions in which \( \text{Sol}_X(\mathcal{D}_X) \) looks like \( \mathcal{C}_X \). So given holomorphic functions on some (open) domain with boundary, there will always be solutions that become singular on the boundary. Phrased another way:
\[
\mathcal{O}_X(X_{\leq 0}) \rightarrow \mathcal{O}_X(X_{< 0}) \text{ is never an isomorphism.}
\]

**Exercise 16.1.** Determine what \( \mu_s(\mathcal{F}) \) is measuring for points \((x_0, 0) \in T^*X\).

16.2. **Holonomic complexes.** Suppose \( \mathcal{M} \in D_{\text{hol}}^b(\mathcal{D}_X) \), complexes of \( \mathcal{D}_X \)-modules with holonomic cohomology sheaves. Recall \( \mathcal{M} \) holonomic means that \( SS(\mathcal{M}) \subseteq T^*X \) is Lagrangian. We can write
\[
SS(\mathcal{M}) := \bigcup_i SS(\mathcal{H}^i(\mathcal{M})) = \Lambda = \bigcup_j \Lambda_j
\]
where \( \Lambda \) is a conic Lagrangian, and the \( \Lambda_j \) are irreducible Lagrangians.

**Example 95.** See Figure 40.

![Figure 40. Decomposition of the singular support into irreducible Lagrangians.](Image)

Define
\[
Y_i = \pi_X(\Lambda_i) = \Lambda_i \cap X,
\]
some closed analytic irreducible subsets of \( X \).

**Lemma 16.5.** \( \Lambda_i = T^*Y_{Y_i \cap X} \).

**Idea:** The \( Y_i \) are giving a stratification on which solutions to \( \mathcal{M} \) are locally constant.
Theorem 16.6. Given a conic Lagrangian $\Lambda$, there exists a Whitney stratification of $X$,

$$X = \bigsqcup_j X_j, \quad X_j \text{ smooth},$$

such that each $Y_i$ is a union of some $X_j$.

Corollary 16.7. $\Lambda \subseteq \bigcup T^*_X X$.

Whitney conditions: These are conditions on the way the normal bundles to $X_j \subset X$ behave (roughly, they have to look locally constant).

Proposition 16.8. If $\mathcal{M}$ is a holonomic $D_X$-module, $\mathcal{F} = \text{Sol}_X(\mathcal{M}) \in D^b(\mathbb{C}X)$, and $\mu_s(\mathcal{F}) = \Lambda$ is a conic Lagrangian, then $\mathcal{F}$ is constructible with regards to $X_j$. I.e. $\mathcal{H}^k(\mathcal{F})|_{X_j}$ is locally constant with finite dimensional fibres.

Figure 41. Rough picture of the Whitney condition.

17. Regular singularities.

We have now seen that if $\mathcal{M}$ is holonomic, then $\text{Sol}_X(\mathcal{M})$ is constructible (locally constant and finite dimensional on each stratum, see Figure 42). We use this fact to motivate a new class of $D$-modules – those with regular singularities.
Recall:

\[ \text{DR}_X(\mathcal{M}) = \mathcal{M} \xrightarrow{\sigma} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_X \rightarrow \cdots \rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^n_X \]

We can identify this with

\[ \text{DR}_X(\mathcal{M}) = \text{RHom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})[n]. \]

How?

- \( \mathcal{O}_X \) has a resolution by free \( \mathcal{D}_X \)-modules

\[ (\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^n_{-n} \mathfrak{T}_X \rightarrow \cdots \rightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^2_{-2} \mathfrak{T}_X \rightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathfrak{T}_X \rightarrow \mathcal{D}_X \xrightarrow{\sigma} \mathcal{O}_X). \]

- So taking homs from this resolution by free \( \mathcal{D}_X \)-modules, we can move polyvector fields from the LHS to its dual (forms) on the RHS. Then to agree with the degrees of \( \text{DR}_X(\mathcal{M}) \) we need to shift back down by \( n \).

Now applying Verdier duality,

\[ \text{RHom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})[n] = \text{RHom}(\mathcal{D}_X, \mathcal{M}) \mathcal{D}_X \mathcal{O}_X)[n] = \text{Sol}_X(\mathcal{D}_X, \mathcal{M}), \]

so,

\[ \text{DR}_X(\mathcal{M}) = \text{Sol}_X(\mathcal{D}_X, \mathcal{M}). \]
Example 96. If $\mathcal{M} = \mathcal{O}_X$, $\text{DR}_X(\mathcal{O}_X) \simeq \mathbb{C}_X$. So we have a functor

$$
\begin{array}{ccc}
D^b_{\text{coh}}(D_X) & \xrightarrow{\text{DR}_X} & D^b(C_X) \\
\downarrow & & \uparrow \\
D^b_{\text{hol}}(D_X) & \text{essentially surjective} & D^b_{\text{c}}(C_X) \\
\downarrow & & \\
D^b_{\text{rs}}(D_X) & & \\
\end{array}
$$

where the bottom most category is the bounded derived category of $D_X$-modules with regular singularities.

Definition 44. A holonomic $D_X$-module $\mathcal{M}$ has regular singularities at $x \in X$ if

$$
\begin{array}{ccc}
\text{Sol}_X(\mathcal{M})_x & \xrightarrow{\sim} & \text{Sol}_X(\mathcal{M})_x \\
\downarrow & & \downarrow \\
\text{Hom}_{D_X, x}(\mathcal{M}, \mathcal{O}_X, \mathbb{C}[[x_1, .., x_n]]) & & \\
\end{array}
$$

“Formal solutions of $\mathcal{M}$.”

I.e. “every formal solution has a positive radius of convergence”.

Warning: This is a definition for analytic and not algebraic $D$-modules.

Then we define $D^b_{\text{rs}}(D_X)$ to be complexes with cohomology objects regular singular $D_X$-modules (i.e. regular singular at every point).

Theorem 17.1 (Riemann-Hilbert Correspondence).

$$
\text{DR}_X : D^b_{\text{rs}}(D_X) \xrightarrow{\sim} D^b_{\text{c}}(C_X)
$$

is an equivalence of triangulated categories.

Remark We no longer have the flat connections $\leftrightarrow \pi_1$-reps correspondence with complexes. Instead we need to look at the exit path category. See, e.g. [AFR].

Example 97. Let

$$
\text{Conn}(X) = \left\{ \begin{array}{c}
\text{vector bundles on } X \\
\text{with flat connections}
\end{array} \right\} \subseteq D^b_{\text{rs}}(D_X).
$$

Then

$$
\begin{array}{ccc}
\text{Conn}(X) & \xrightarrow{\sim} & \{ \text{locally constant sheaves} \} \\
(\mathcal{V}, \nabla) & \xrightarrow{\sim} & \text{DR}_X(\mathcal{V}, \nabla) \\
(\mathcal{F} \otimes_{\mathbb{C}_X} \mathcal{O}_X, d) & \xrightarrow{\sim} & \mathcal{F}[n]
\end{array}
$$

There is a Poincaré lemma that says $\text{DR}_X(\mathcal{V}, \nabla)$ has no higher cohomology – it is all concentrated in degree $-n$, thus

$$
\text{DR}_X(\mathcal{V}, \nabla) \simeq \ker(\nabla)[n].
$$

Remark The Riemann-Hilbert correspondence was proven independently by Mebkhout (1979) and Kashiwara (1980).

17.1. What does “regular singularities” mean?

Example 98. Let

$$
\mathcal{M} = D_X / D_X \cdot P \simeq (D_X \xrightarrow{P} D_X),
$$
(we have an easy free resolution in this case). So
\[ \text{Sol}_X(\mathcal{M}) \cong \mathcal{O}_X \xrightarrow{p(-)} \mathcal{O}_X. \]

Hence
\[ \mathcal{H}^0\text{Sol}_X(\mathcal{M}) = \ker(P) \]
\[ \mathcal{H}^1\text{Sol}_X(\mathcal{M}) = \coker(P) \]

We can localize everything to look at germs of solutions around \( x \in X \),
\[ \text{Sol}_X(\mathcal{M})_x \cong \mathcal{O}_{X,x} \xrightarrow{p(-)} \mathcal{O}_{X,x}. \]
\[ \mathcal{H}^0\text{Sol}_X(\mathcal{M})_x = \ker P \]
\[ \mathcal{H}^1\text{Sol}_X(\mathcal{M})_x = \coker P \]

Formal solutions are:
\[ \hat{\text{Sol}}_X(\mathcal{M})_x = \hat{\text{Sol}}_x(\mathcal{M}) = \hat{\mathcal{O}}_{X,x} \xrightarrow{p} \hat{\mathcal{O}}_{X,x}. \]

Assume \( X \) is 1d (so that \( \mathcal{M} \) can be holonomic), and since we are working locally, without loss of generality let \( X = \mathbb{C} \). Then
\[ \hat{\mathcal{O}}_{X,x} = \hat{\mathcal{O}} = \mathbb{C}\{x\} \]
\[ \hat{\mathcal{O}}_{X,x} = \hat{\mathcal{O}} = \mathbb{C}[x]. \]

**Example 99.** Let \( P = x\partial_x - \lambda \). Since \( x\partial_x(x^n) = nx^n \), \( x^n \) is an eigenvalue for \( x\partial_x \). So the solution to \( Pu = 0 \) is \( u = x^\lambda \) – if \( \lambda \) is non-integral this requires a choice of branch of log (solution only makes sense on some small simply connected region). How can we think of this?

\[
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
x^4 & 4-\lambda & x^4 & \\
x^3 & 3-\lambda & x^3 & \\
x^2 & 2-\lambda & x^2 & \\
x & 1-\lambda & x & \\
1 & -\lambda & 1 & \\
\end{array}
\xrightarrow{P} \]

How do the kernel and cokernel compare for power series versus formal power series?

**Case 1:** \( \lambda \notin \mathbb{Z}_{\geq 0} \); then all of the maps above are isomorphisms, so
\[ \text{Sol}_X(\mathcal{M})_x \cong \hat{\text{Sol}}_X(\mathcal{M})_x \cong 0. \]

**Case 2:** \( n = \lambda \in \mathbb{Z}_{\geq 0} \). Then \( x^n \) is in the kernel of \( P \), and in fact
\[ \ker(P) = \mathbb{C}x^n. \]

Also, \( x^n \) is not in the image of \( P \), so
\[ \coker(P) \cong \mathbb{C}x^n. \]

This holds for \( \mathbb{C}\{x\} \) and \( \mathbb{C}[x] \). Thus,
\[ \mathcal{D}/\mathcal{D} \cdot (x\partial_x - \lambda) = \mathcal{D} \cdot x^\lambda \]

is regular for every \( \lambda \).

**Example 100.** \( P = x^2\partial_x - 1 \). \( e^{x^2} \) is a solution – this has an essential singularity in the analytic sense. Looking at this:
• **Observation 1:** $\ker(P) = \ker(\hat{P}) = 0$. Why? $e^x$ is the unique solution (to this first order ODE), but it is not holomorphic at 0. So $\ker(P) = 0$.

• **Observation 2:** $\hat{P}$ is an isomorphism.

• **Observation 3:** $P$ is not surjective.

For $\hat{P}$:

$$\hat{P}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) = -a_0 + (a_1 x^2 - a_1 x) + (2a_2 x^3 - a_2 x^2) + (3a_3 x^4 - a_3 x^3) + \cdots$$

$$= -a_0 - a_1 x + (a_1 - a_2) x^2 + (2a_2 - a_3) x^3 + \cdots$$

Thus $\ker(\hat{P}) = 0$ (solve the above equation term-by-term from $a_0$).

Moreover, given a power series we can algebraically recursively solve for the values of $a_0, a_1, \ldots$. E.g. there exist $a_0, a_1, \ldots$ such that $\hat{P}(a_0 + a_1 x + a_2 x^2 + \cdots) = x$.

Here we need

$$0 + a_0 x + (-1)x^2 - 2x^3 + \cdots \quad \text{(solving recursively).}$$

It is not hard to see that this will have 0 radius of convergence.

**Conclusion:** $\hat{P}$ is an isomorphism, so $\hat{\Sol}_X(\mathcal{M}) = 0$.

But $\Sol_X(\mathcal{M})$ turns out to have cokernel generated by $x$; so

$$\Sol_X(\mathcal{M}) \cong \mathbb{C}[-1].$$

Thus,

$$\mathcal{M} = \mathcal{D}/\mathcal{D} \cdot (x^2 \partial_x - 1)$$

is irregular at 0.

Continue assuming $X$ is 1d, and since we are working locally, continue assuming $X = \mathbb{C}$ and $x = 0$.

**Question:** When is $\mathcal{D}/\mathcal{D} \cdot P$ regular?

**Definition 45.** $\delta(P) = m - \text{ord}_0(a_m)$, where

$$P = a_m(x) \partial + x^m + a_{m-1}(x) \partial_{x}^{m-1} + \cdots + a_0(x),$$

and $\text{ord}_0$ is the order of vanishing at 0.

We can think of $\mathcal{D}$ as $\mathcal{D} = \mathcal{O} [\partial]$.

**Definition 46.**

$$\hat{\delta}(P) = \max_{0 \leq k \leq m} \{ k - \text{ord}_0(a_k) \}.$$

By convention, $\text{ord}_0(0) = +\infty$.

**Example 101.** If $P = x^N \partial^m$, then $\delta(P) = \hat{\delta}(P) = m - N$.

**Definition 47.** The *index* of $P$ is

$$\chi(\mathcal{D}/\mathcal{D} \cdot P) := \chi(\Sol_X(\mathcal{D}/\mathcal{D} \cdot P)_0) \quad \text{(which is a two term complex),}$$

$$= \dim(\ker(P)) - \dim(\coker(P)).$$

Similarly we define

$$\hat{\chi}(\mathcal{D}/\mathcal{D} \cdot P) = \chi(\hat{\Sol}_X(\mathcal{D}/\mathcal{D} \cdot P)).$$

**Exercise 17.1.** Show that $\mathcal{D}/\mathcal{D} \cdot P$ is regular if and only if $\chi(\mathcal{D}/\mathcal{D} \cdot P) = \hat{\chi}(\mathcal{D}/\mathcal{D} \cdot P)$.

**Theorem 17.2 (Index Theorem).**

$$\delta(P) = \chi(\mathcal{D}/\mathcal{D} \cdot P)$$

$$\hat{\delta}(P) = \hat{\chi}(\mathcal{D}/\mathcal{D} \cdot P)$$
Corollary 17.3. $\mathcal{D}/\mathcal{D} \cdot P$ is regular if and only if $\delta(P) = \hat{\delta}(P)$.

17.2. Regular singularities in the local setting. Recall the notation:

\( \mathcal{O} := \mathbb{C}\{ \{ x \} \} \cong (\mathcal{O}_\mathbb{C})_0 \supset M = x\mathcal{O}, \) and \( \hat{\mathcal{O}} := \mathbb{C}[ [ x ] ] \).

Remark \( \mathbb{C}[ x ] \subset \mathbb{C}\{ \{ x \} \} \subset \mathbb{C}[ [ x ] ] \).

Define the field of fractions of \( \mathcal{O} \) to be

\( \mathcal{K} := \mathcal{O}[ x^{-1} ], \)

(i.e. local meromorphic functions).

Example 102. For \( \mathcal{D}/\mathcal{D} \cdot P \) a \( \mathcal{D} \)-module,

\[
\text{Sol}(\mathcal{D}/\mathcal{D} \cdot P) = \mathcal{O} \xrightarrow{P} \mathcal{O}, \\
\widehat{\text{Sol}}(\mathcal{D}/\mathcal{D} \cdot P) = \hat{\mathcal{O}} \xrightarrow{\hat{P}} \hat{\mathcal{O}}.
\]

Notation convention: From now on,

\( \text{Sol}(P) := \text{Sol}(\mathcal{D}/\mathcal{D} \cdot P) \).

This is a two term complex, so we only have

\[
\mathcal{K}^0(\text{Sol}(P)) = \ker(P) \\
\mathcal{K}^1(\text{Sol}(P)) = \coker(P)
\]

Remark \( \ker(P) \subset \ker(\hat{P}) \) and \( \coker(P) \rightarrow \coker(\hat{P}) \).

Recall we defined \( \mathcal{D}/\mathcal{D} \cdot P \) to be regular if

\( \text{Sol}(P) \xrightarrow{\sim} \widehat{\text{Sol}}(P) \) (quasi-isomorphism).

This turns out to be equivalent to

\( \chi(P) = \hat{\chi}(P) \).

Remark Our previous remark tells us that we always have \( \chi(P) \leq \hat{\chi}(P) \).

Let

\[
P = \sum_{k=0}^{m} a_k(x) \partial^k, \quad a_m(x) \neq 0.
\]

Then we defined

\[
\delta(P) = m - \text{ord}(a_m(x)) \\
\hat{\delta}(P) = \max_{0 \leq k \leq m} (k - \text{ord}(a_k(x))]
\]

Theorem 17.4. We have that

\[
\delta(P) = \chi(P) \quad \text{and} \quad \hat{\delta}(P) = \hat{\chi}(P).
\]

Proof. The proof for \( \delta \) uses analysis. The idea is that \( P - a_m(x) \partial^m \) is a compact operator between certain Banach spaces (recall that \( P \) is Fredholm). Then there is an index theorem that says \( \chi(P) = \chi(a_m(x) \partial^m) \).

See [B] for details.

The proof that \( \hat{\delta}(P) = \hat{\chi}(P) \) is a purely algebraic computation in manipulating power series. \( \square \)
Claim: $\delta(P) = \tilde{\delta}(P)$ if and only if
\[
P = \sum_{i=0}^{m} b_i(x) \theta^i, \quad b_i(x) \in \mathcal{K}, \quad \theta = x\partial,
\]
and
\[
\frac{b_i(x)}{b_m(x)} \text{ is holomorphic.}
\]

Why? Start with $a_m(x) = x^{m-\delta(P)} \tilde{a}_m(x)$, $\tilde{a}_m(x)$ nonvanishing. Observe that
\[
x^m \partial^m = \theta(\theta - 1) \cdots (\theta - (m - 1)).
\]

Example 103. $P_1 = x\partial - \lambda = \theta - \lambda$ is regular.

Example 104. $P_2 = x^2\partial - \lambda$, $\delta(P_2) = -1 \neq 0$ is not regular. What fails in this example?
\[
P_2 = x(x\partial) - \lambda = x\theta - \lambda,
\]
and $\frac{\lambda}{x}$ is not holomorphic.

Remark When we rewrite $P$ in this form, $\text{ord}(b_i(x)) = \text{ord}(a_k) - k$.

Definition 48. A $P$ of the type
\[
P = \sum_{i=0}^{m} b_i(x) \theta^i, \quad b_i(x) \in \mathcal{K}, \quad \theta = x\partial,
\]
with
\[
\frac{b_i(x)}{b_m(x)} \text{ is holomorphic,}
\]
is called Fuchsian.

Remark Locally, Fuchsian differential operators are regular. Globally, however, they may not be regular – this is in fact a failure of Hilbert’s 23rd problem!

17.3. Another approach (still in the local setting).

(1) $Pu = 0$ an ODE.
(2) $\frac{d}{dx} \bar{u}(x) = \frac{\Gamma(x)}{x} \bar{u}(x)$, where
\[
\Gamma(x) = \begin{pmatrix}
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 1 & \vdots & \vdots \\
\vdots & \vdots & \ddots & 1 & 0 \\
-b_0 & -b_1 & \cdots & \cdots & -b_{m-1} \\
b_0 & b_1 & \cdots & \cdots & b_m
\end{pmatrix}
\]
is a matrix in rational canonical form.

Then solutions $u$ of the first equation are in one-to-one correspondence with solutions of the second equation
\[
\bar{u}(x) = \begin{pmatrix}
u(x) \\
\theta u(x) \\
\vdots \\
\theta^{m-1} u(x)
\end{pmatrix}.
\]
The condition now becomes

\[ \text{P is Fuchsian if and only if } \Gamma(x) \text{ is holomorphic.} \]

18. Meromorphic connections.

**Definition 49.** A meromorphic connection is a finite dimensional \( \mathcal{K} \)-vector space \( \mathcal{M} \), together with \( \nabla : \mathcal{M} \to \mathcal{M} \) satisfying the Leibniz rule

\[ \nabla(fm) = \frac{df}{dx}m + f \nabla m, \quad f \in \mathcal{K}, \quad m \in \mathcal{M}. \]

Let \( \tilde{\mathcal{K}} \) denote the space of (possibly multivalued) holomorphic functions on \( \Delta^* = \{0 < |x| < \epsilon\} \) for arbitrarily small \( \epsilon \). “Possibly multivalued” should be interpreted as functions on the universal cover.

This generalizes the space of meromorphic functions in two ways:

- We are allowing multivalued functions.
- We are allowing essential singularities at 0.

We will use this as a solution space, after a remark on meromorphic connections.

**Remark** If \( \mathcal{M} \) is a meromorphic connection, pick a \( \mathcal{K} \)-basis \( e_1, \ldots, e_n \) of \( \tilde{\mathcal{K}} \). Then we can write

\[ \nabla = d - A, \quad d \text{ the de Rham differential, } A \text{ a matrix.} \]

So horizontal sections \( \tilde{u} \) of \( \mathcal{M} \) are the same as solutions to the equation

\[ \frac{d}{dx} \tilde{u}(x) = A(x) \cdot \tilde{u}(x). \]

**Fact:** If \( \mathcal{M} \cong \mathcal{K}^n \) is a meromorphic connection, then is has a complex \( m \)-dimensional space of solutions in \( \tilde{\mathcal{K}} \).

If \( \tilde{u}_1(x), \ldots, \tilde{u}_m(x) \in \tilde{\mathcal{K}} \) is a basis of horizontal sections, then we say that

\[ S(x) := \left( \begin{array}{c} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_m \end{array} \right) \]

is a fundamental solution.

**Definition 50.** We say that \( f \in \tilde{\mathcal{K}} \) has moderate growth if for all open \( (a,b) \subseteq \mathbb{R} \) and \( \epsilon > 0 \) such that \( f \) is defined on \( S_{(a,b)}^\epsilon \) as shown in Figure 43, there exists \( N >> 0 \) such that

\[ |f(x)| \leq C|x|^{-N} \quad \text{for all } x \in S_{(a,b)}^\epsilon. \]

**Note:** \( \mathcal{M} \) is a \( \mathcal{D} \)-module via \( \nabla \leftrightarrow \partial \).

![Figure 43. Universal cover of punctured disk, with strip \( S_{(a,b)}^\epsilon \) shown.](image)
Fact: If $f$ is single valued, then moderate growth is equivalent to meromorphy. (I.e. this is the way we say a multivalued function does not have essential singularities.)

Theorem 18.1. For $\mathcal{M}$ a fixed meromorphic connection, the following are equivalent:

1. $\mathcal{M}$ is equivalent to a system of the form,
   \[ \frac{d}{dx} \bar{u}(x) = \Gamma(x) \bar{u}(x), \]
   where $\Gamma(x) \in M_n(\mathcal{O})$ ($n \times n$ matrices).
2. Same as above, but $\Gamma$ is a constant matrix.
3. All solutions to $\mathcal{M}$ in $\tilde{\mathcal{K}}^m$ have moderate growth.

Also, all of these are equivalent to $\mathcal{M}$ being regular.

Sketch that (3) implies (2). Let $S(x)$ be a fundamental solution matrix. Take
   \[ \lim_{t \to 1} S(e^{2\pi i} x) = G \cdot S(x); \]
the LHS is another fundamental solution, so can be expressed in the form of the RHS. Call $G$ the monodromy matrix. Let $\Gamma$ be some matrix such that
   \[ e^{2\pi i \Gamma} = G, \]
where we can use the Jordan form for $G \in GL_n \mathbb{C}$ to make finding such a $\Gamma$ easier. Note that $e^{\Gamma \log(x)}$ has the same monodromy as $S(x)$. So define
   \[ T(x) := S(x) \cdot e^{-\Gamma \log(x)}. \]
Then $T(x)$ is single valued with moderate growth (our original assumption), so $T(x)$ is meromorphic. Then $\tilde{u}(x)$ is a solution of $\mathcal{M}$ if and only if $T(x)^{-1} \tilde{u}(x)$ is a solution of
   \[ \frac{d}{dx} (-) = \Gamma(x) (-) \]
where $\Gamma$ is a constant matrix. \qed

Remark For those who are interested, the key ingredient in (1) $\Rightarrow$ (2) is Grönwall’s inequality. We will not be discussing this.

Example 105. (1) $\frac{d}{dx} u(x) = \frac{\lambda}{x} u(x)$ has solutions $u(x) = x^\lambda$. This is a multivalued function: to make sense of it, consider choosing a branch of log, of considering it as a function on the universal cover of $\Delta^*$. This is regular.
(2) $\frac{d}{dx} u(x) = -\frac{1}{x^2} u(x)$ has as solution the single-valued function $u(x) = e^{\frac{1}{x}}$. This is non-regular $- e^{\frac{1}{x}}$ can grow faster than any meromorphic function.

Corollary 18.2. If $Pu = 0$ is an ODE (so $P \in \mathcal{D}$ – still considering the local situation) then $P$ is Fuchsian if and only if all solutions to $P$ in $\tilde{\mathcal{K}}$ have moderate growth.

Definition 51 (/Proposition). A meromorphic connection is regular if there exists a finitely generated $\mathcal{O}$-submodule $L \subseteq \mathcal{M}$ such that
   \[ \Theta L \subseteq L, \ \Theta = x \partial, \]
and $L$ generated $\mathcal{M}$ over $\mathcal{K}$. We call $L$ a lattice.

Remark This gives a coordinate free expression of regularity.

Given $\mathcal{M}$ as in the theorem, take a basis $e_1, \ldots, e_m$ of $\mathcal{M}$ such that the connection matrix looks like
   \[ \frac{\Gamma(x)}{x}, \ \Gamma(x) \in M_n(\mathcal{O}). \]
Then take $L = \Theta e_1 + \cdots + \Theta e_m$. 

Lemma 18.3. Given \( L \) as in the definition/proposition, we have that 
\[ L \cong \mathcal{O}^m \quad (L \text{ is free}). \]
Thus we can think of \( L \) as a vector bundle on the unpunctured disk with connection 
\[ \nabla : L \to L \otimes \Omega^1(d\log(x)) \]
induced from the original connection on \( \mathcal{M} \).

I.e. \( L \) is a vector bundle with connection that has logarithmic poles. Here 
\[ d\log(x) = \frac{dx}{x}. \]

Example 106. From the first example above, 
\[ \mathcal{M} = \mathcal{K} \quad \text{and} \quad \nabla = d - \frac{\lambda}{x} dx. \]
We already (implicitly) chose a trivialization \( e \in \mathcal{M} \) to write this, and then \( L = \mathcal{O} e \subseteq \mathcal{M} \).

19. **Global theory of regular singularities.**

There are two directions in which we could generalize this topic:

- Irregular connections. (Unfortunately we won’t have time for this.)
- Global theory. (This will be our focus.)

**Hilbert’s 21st problem:** Let \( \{a_1, \ldots, a_k\} \subseteq \mathbb{A}^1 = \mathbb{C} \). Given \( G_1, \ldots, G_k \in GL_n\mathbb{C} \), does there exist a Fuchsian differential equation with singularities at \( \{a_i\} \) and monodromy \( G_i \) at \( a_i \)?

**Remark** In the local case, given \( G \in GL_n\mathbb{C} \) we can always find \( \Gamma \in M_n(\mathbb{C}) = gl_n \) such that 
\[ e^{2\pi i \Gamma} = G \quad \text{----->} \quad \frac{d}{dx} \tilde{u}(x) = \Gamma \frac{x}{x} \tilde{u}(x). \]

Note that:

1. \( \pi_1(\mathbb{A}^1 - \{a_1, \ldots, a_k\}) \cong F_k \), the free group on \( k \) letters.

\[ \text{Figure 44. Determining } \pi_1(U). \]

So,
\[ \{\text{Local systems on } U\} \simeq \text{Rep}(\pi_1(U)) \cong GL_n(\mathbb{C})^k/GL_n(\mathbb{C}), \]
i.e. a choice of \( k \) matrices up to simultaneous conjugation (change of basis).

2. \( \mathbb{A}^1 = \mathbb{C} \subseteq \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \cong S^2 \). So we could also demand in Hilbert’s 21st problem that \( \infty \) is not a singular point. For a long time, people thought that this problem would have a positive answer – in fact it does **not**, but something “very close” does.
19.1. Deligne’s “solution” to Hilbert’s problem.

First idea:

\[ \text{Rep}(\pi_1(U)) \leftrightarrow \left\{ \text{Locally constant sheaves on } U \right\} \leftrightarrow \{ \text{Flat connections on } U \} \]

\[ \mathcal{L} \leftrightarrow (\mathcal{O}_X \otimes_{\mathbb{C}_X} \mathcal{L}, d) \]

Given a flat connections \((\mathcal{V}^*, \nabla^*)\) on \(U\), Deligne defined an extension \(\mathcal{M}\) a meromorphic flat connection on \((\mathbb{P}^1, \{a_1, \ldots, a_k\})\). I.e. \(\mathcal{M}\) is a locally free \(\mathcal{O}_{\mathbb{P}^1} \left[ \frac{1}{x-a_1}, \ldots, \frac{1}{x-a_k} \right]\)-module such that \(\mathcal{M}|_U \cong \mathcal{V}^*\).

Note: For \(j : U \hookrightarrow \mathbb{P}^1\) you can always take

\[ j_*(\mathcal{V}^*) \in \mathcal{O}_{\mathbb{P}^1}\)-modules.

But in the world of analytic geometry, this sheaf allows for essential singularities, so is not a meromorphic connection. (This cannot happen in the algebraic world.) The \(\mathcal{M}\) we construct will be a subsheaf

\[ \mathcal{M} \subseteq j_*(\mathcal{V}^*). \]

Turns out: \(\mathcal{M}\) has regular singularities at \(a_1, \ldots, a_k\); i.e. there is \(L \subseteq \mathcal{M}\) a vector bundle on \(\mathbb{P}^1\) such that \(\nabla\) has log poles with respect to \(L\).

Note: \(L\) may not be trivial! So this doesn’t necessarily correspond to what one might think of as a system of differential equations on the plane.

I.e. This does not imply that Hilbert’s 21st problem is true as stated above. But you could consider this the “correct” version (or “corrected” version) of the problem. Why? Because a differential equation is equivalent to

\[(\mathcal{M}, L, \nabla)\] together with a trivialization \(L \cong \mathcal{O}^m\).

But if \(L\) is not trivial, then the trivialization does not exist.

Remark \(\mathcal{M}\) is a \(D\)-module: take its de Rham complex,

\[ \text{DR}_{\mathbb{P}^1}(\mathcal{M}) = Rj_*\left( \text{DR}_U(\mathcal{V}^*) \right). \]

19.2. Higher dimensions. If \(X\) is a complex manifold, \(D \subseteq X\) a hypersurface, then Deligne proved

\[ \text{Conn}^{\text{reg}}(X, D) \leftrightarrow \text{Conn}(U) \]

\[ \text{flat connections on } X, \text{ meromorphic along } D, \text{ with regular singularities} \]

\[ \text{all flat connections on } U = X - D \]

We call this Deligne’s Riemann-Hilbert correspondence.

20. Algebraic story.

As far as Sam knows, there is no self-contained algebraic story – it has to pass through the analytic story.

Suppose \(X\) is projective, \(X \subseteq \mathbb{P}^N\). Then \(X\) is an algebraic variety (Chow’s theorem). GAGA then implies that

\[ \text{Conn}^{\text{reg}}(X^{\text{alg}}, D^{\text{alg}}) \simeq \text{Conn}^{\text{reg}}(X^{\text{an}}, D^{\text{an}}) \simeq \text{Conn}(U^{\text{an}}). \]

Of course we can also restrict

\[ \text{Conn}^{\text{reg}}(X^{\text{alg}}, D^{\text{alg}}) \xrightarrow{\text{(defn)}} \text{Conn}^{\text{reg}}(U^{\text{alg}}) \subset \text{Conn}(U^{\text{alg}}). \]
Example 107. On $\mathbb{A}^1 = \mathbb{C} = U$ define

$$\mathcal{V}^* = (\mathcal{O}_U, d - \lambda) \leftrightarrow \text{solutions} \rightarrow e^{\lambda x}.$$  

$e^{\lambda x}$ is analytic and not algebraic (for $\lambda \neq 0$). But it is the solution to an algebraic equation. So $(\mathcal{O}_U, d - \lambda)$ is an algebraic $\mathcal{D}$-module on $\mathbb{A}^1$,

$$\mathcal{D}_U/\mathcal{D}_U(\partial - \lambda),$$

and these are all inequivalent for different $\lambda$. Analytically, however, $(\mathcal{V}^*)^{an} \simeq (\mathcal{O}_X^{an}, d)$ (i.e. we have equivalence for all $\lambda$).

Fact: This algebraic $\mathcal{D}$-module is not regular unless $\lambda = 0$.

Why? Need to embed $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$. Then $e^{\lambda x} \mapsto e^{\frac{\lambda}{w}}$ near infinity, which has an essential singularity.

Note that $\mathcal{M}^{alg} = j^{alg*}(\mathcal{V}^*)$ is a meromorphic connection, but is not regular unless $\lambda = 0$.


We work in the analytic setting. Let $X$ be a complex manifold, $D \subseteq X$ a complex hypersurface, $U = X - D$. Consider Figure 45:

![Figure 45](image-url)

Figure 45. The Riemann-Hilbert correspondence.

Claim that this diagram in Figure 45 commutes.

Example 108. Some possible choices of $(X, D)$:

(1) Punctured Riemann surface as in Figure 46.
(2) $Z \subseteq X$ a closed subvariety, $E \subseteq Z$ a hypersurface such that $Z - E =: V$ is smooth – see Figure 47.

Let $k : V \hookrightarrow Z$, $i : Z \hookrightarrow X$. Given $\mathcal{L}$ a local system on $V$, we produce

$$i_* Rk_*(\mathcal{L}) \in D^b_c(X).$$

In fact, these generate the category $D^b_c(X)$ as we range over all possibilities of $(Z, E)$.

We have an even more refined picture of the Riemann-Hilbert correspondence in Figure 48:
We can define \( \text{Perv}(X) \) to be the image of \((\mathcal{D}\text{-mod})_h\) under Riemann-Hilbert, so that the equivalence in Figure 48 is tautological. But we can give an intrinsic characterisation as well.

**Definition 52.** An object \( \mathcal{F} \in D^b_c(X) \) is called a **perverse sheaf** if

- \( \dim(\text{supp}(\mathcal{H}^j(\mathcal{F}))) \leq -j \), and
- \( \dim(\text{supp}(\mathcal{H}^j(\mathcal{D}X\mathcal{F}))) \leq -j \).

**Remark** For \( X \) a \( \mathbb{C} \)-manifold with \( \dim \mathbb{C} X = n \),

\[
\mathbb{D}_X(\mathcal{F}) = R\mathcal{H}om_{\mathbb{C}X}(\mathcal{F}, \omega_X).
\]

**Example 109.** \( \mathbb{C}X[n] \in \text{Perv}(X) \). (Recall: \( \mathbb{C}X[n] \cong \text{DR}_X(\mathcal{O}_X) \).)

- \( \mathbb{D}_X(\mathbb{C}X[n]) = R\mathcal{H}om(\mathbb{C}X[n], \mathbb{C}X[2n]) = \mathbb{C}X[n] \).
- \( \dim(\text{supp}(\mathcal{H}^{-n}(\mathbb{C}X[n]))) = n \).

**Example 110.** Let \( i : Z \subseteq X \) be a closed submanifold. Then

\[
i_*\mathbb{C}Z[\dim Z] \in \text{Perv}(X).
\]

**Remark** All our shifts make it so that Poincaré duality is a “symmetric flip” across degree 0, rather than a “shifted flip”.

**Fact:** For irreducible \( \mathcal{L} \in \text{Loc}(U)[n] \), there is a unique irreducible subobject of

\[
Rj_*(\mathcal{L}) \in \text{Perv}(X).
\]

We call it \( IC(X, \mathcal{L}) \in \text{Perv}(X) \), the **minimal extension/intermediate extension/Goresky-MacPherson extension**. \( IC \) stands for **intersection cohomology**. This has the property that

\[
IC(X, \mathcal{L})|_U \cong \mathcal{L}.
\]

What does this correspond to in the \( \mathcal{D} \)-module world?

\[
\mathcal{L} \in \text{Loc}(U) \leftrightarrow \mathcal{M} \in \text{Conn}^{\text{reg}}(X, D),
\]
TYPED BY RICHARD HUGHES FROM LECTURES BY SAM GUNNINGHAM

and there is a unique choice of lattice $L \subseteq \mathcal{M}$ such that the eigenvalues of $\Theta = f \partial f$ have real part in $[0, 1)$ (a fundamental domain for the exponential function). Then

$$IC(X, \mathcal{L}) \leftrightarrow \mathcal{D}_X \cdot L \subseteq \mathcal{M},$$

the $\mathcal{D}$-module generated by $L$.

**Remark** Perv($X$) is an artinian abelian category. So *irreducible* means no nonzero subobjects, and such an object exists by the artinian property.

22. **WIDER CONTEXT IN MATHS.**

Want to understand the topology of complex varieties.

**Example 111.** $X \xrightarrow{f} C$ a flat proper family of varieties over a curve as in Figure 49:

![Figure 49](image)

**Figure 49.** A flat proper family of varieties $X$ over a curve $C$.

Under the Riemann-Hilbert correspondence we have

$$f^\text{dR}_* (\mathcal{O}_X) \leftrightarrow Rf_*(\mathbb{C}_X) \in \text{D}^b_c (C).$$

Generically on $C$, this is a vector bundle with a flat connection (the *Gauss-Manin connection*), and we have

$$R^i f_*(\mathbb{C}_X)_t = H^i(X_t; \mathbb{C}).$$

**Example 112.** Take the family

$$X = \{ y^2 = x(x - 1)(x - t) \} \xrightarrow{f} (\mathbb{P}^1, \{ 0, 1, \infty \})$$
whose fibre over \( t \) are the solutions to the given equation. Then

\[ \mathcal{R}^1 f_*(\mathcal{C}_X) \longleftrightarrow \text{Picard-Fuchs equation on } \mathbb{P}^1. \]

It is a classical result that this equation has regular singularities at \( \{0, 1, \infty\} \). Solutions of this differential equation are hypergeometric functions.

If \( X \) is a compact Kähler manifold, then

\[ H^i(X; \mathbb{C}) = \bigoplus_{p+q=i} H^q(X; \Omega^p_X), \]

This is called a Hodge structure. What happens to this structure as we vary \( X \) in a family?

**Remark** The Hodge structure also tells us how \( H^i(X; \mathbb{Z}) \subseteq H^i(X; \mathbb{C}) \) intersects with the decomposition.

**Theorem 22.1** (A Torelli Theorem). If \( X \) is an elliptic curve, then \( X \) is determined by \( H^1(X) \) with Hodge structure:

\[ H^1(X; \mathbb{Z}) \xrightarrow{\mathbb{Z}^2} H^0.1(X) \xrightarrow{\text{quotient}} X \]

**Hodge filtration:** (Warning: Check that the \( p \)'s and \( q \)'s are correct below!)

\[ F^p H^i(X, \mathbb{C}) = \bigoplus_{p' \geq p, q=0} H^{p',q}(X). \]

As the fibres \( X_t \) vary with \( t \), the \( F^p H^i(X_t) \) form a holomorphic subbundle of the corresponding flat connection on, e.g., \( \mathbb{P}^1 - \{0, 1, \infty\} \) (i.e. on the smooth locus).

This leads to the notion of a variation of Hodge structure,

\[ \nabla \downarrow \quad \nabla \downarrow \quad F^\bullet \uparrow \quad V_{\mathbb{Z}} \quad \alpha : \ker(\nabla) \xrightarrow{\sim} (V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C})), \]

\[ \quad \quad \quad \text{vector bundle} \quad \text{locally const.} \]

\[ \quad \quad \quad \text{w/ flat conn.} \quad \text{sheaf of free} \]

\[ \quad \quad \quad \text{filtration of } \mathcal{V} \quad \text{abelian groups} \]

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The Griffiths transversality theorem: \( \nabla F^i \subseteq F^{i-1} \).

Then we have an analogy

\[ \mathcal{D}\text{-module} \quad : \quad \text{vector bundle with flat connection} \]

\[ \mathcal{D}\text{-module} \quad : \quad \text{variation of Hodge structure} \]

**22.1. Further topics.**

- Mixed Hodge modules (nearby/vanishing cycles)
- Moduli of flat connections/Moduli of Higgs bundles (Geometric Langlands)
- Beilinson-Bernstein (e.g. representations of \( \mathfrak{sl}_2 \) with trivial central character correspond to \( \mathcal{D}\text{-modules on } \mathbb{P}^1 \)
References

Figure 36. The Riemann-Hilbert correspondence.