Abstract. In this talk, I will introduce and motivate the main objects of study in modular representation theory, which leads one to consider the stable module category. Perhaps surprisingly, one can do homotopy theory in this setting, as the stable module category has the structure of a stable model category. Therefore, there is a very rich interplay between modular representation theory and stable homotopy theory, which I will try to illustrate. In the spirit of Halloween, one of the ideas I will talk about is the notion of phantom maps.

1. (Modular) Representation Theory

For the entirety of this talk, let $k$ be a field of positive characteristic $p$, and let $G$ be a finite group such that $p 
mid |G|$. We would like to look at the representations $\hat{\rho} : G \to \text{End}_k(V)$. We can then extend this to a group algebra representation $\rho : kG \to \text{End}_k(V)$, which is the same as looking at modules over the group algebra $kG$. This is why we consider the category of modules over $kG$.

Why is this different from ordinary/characteristic 0 representation theory? For one thing, Maschke’s theorem does not apply:

**Theorem 1.1** (Maschke). The group algebra $kG$ is semisimple (Cartesian product of simple subalgebras) iff the characteristic of $k$ does not divide the order of $G$.

**Remark 1.2.** The proof involves dividing by $|G|$.

This in particular tells us things about projective modules:

**Proposition 1.3.** The following are equivalent definitions for a module $P$ being projective:

(i) For every surjective map $M \to N$ and every map $P \to N$, there exists a map from $P \to M$.

(ii) Every short exact sequence of the form $0 \to A \to B \to P \to 0$ splits.

(iii) There is a free module $F$ and a module $Q$ such that $P \oplus Q = F$.

(iv) Hom$(P, -)$ is an exact functor. (This functor is always left exact, but it is also right exact iff $P$ is projective.)

If we are in a situation where we can use Maschke’s theorem, then we can prove that every finitely generated $kG$-module is completely reducible. This, combined with a theorem attributed to Wedderburn (in Dummit & Foote), which gives us the following proposition:

**Proposition 1.4.** Every $R$-module is projective iff every $R$-module is completely reducible.

1.1. But why projective modules?

One reason to consider projective modules is for calculating group cohomology $H^*(G; k)$, and Ext$_{kG}$ groups in general. To do so, we make the following definition:

**Definition 1.5.** If $M \in kG$-Mod, a projective resolution $P_* \to M$ is an exact sequence of projective modules:

$$\cdots \to P_n \to \cdots \to P_0 \to M$$
If $M, N \in kG$-Mod, then we first take a projective resolution $P_\ast \to M$. Then we apply the functor $\text{Hom}(-, N)$ to the sequence to get a complex $\text{Hom}(P_\ast, N)$.

\[ 0 \to \text{Hom}(P_0, N) \to \text{Hom}(P_1, N) \to \cdots \]

**Definition 1.6.** We define $\text{Ext}^n_{kG}(M, N)$ as the homology groups of the complex $\text{Hom}(P_\ast, N)$:

\[ \text{Ext}^n_{kG}(M, N) := H^n(\text{Hom}(P_\ast, N)) \]

**Example 1.7.** If we take $M = k$, the trivial module, then we get group cohomology.

\[ \text{Ext}^n_{kG}(k, N) := H^n(G; k) \]

The use of projective resolutions is just a fibrant replacement and a way to calculate the right derived functors of $\text{Hom}(-, N)$.

We now consider a very nice choice of projective resolution:

**Definition 1.8.** Given $M$ in $kG$-Mod, a projective cover of $M$ is a projective module with a map $\widetilde{P}_M \to M$ such that if $Q \to M$ is morphism from projective, then exists an injective homomorphism $\widetilde{P}_M \to Q$.

**Proposition 1.9.** Projective covers exist and are unique for all $M$ in $kG$-Mod. (Projective covers do not necessarily exist for arbitrary rings $R$).

**Definition 1.10.** If $M \in kG$-Mod, a minimal projective resolution of $M$ is the projective resolution obtained by taking successive projective covers:

\[ \cdots \to \widetilde{P}_n \xrightarrow{\delta_n} \widetilde{P}_{n-1} \xrightarrow{\delta_{n-1}} \cdots \to \widetilde{P}_0 \xrightarrow{\delta_0} M \]

This allows one to simplify computations, but we will also use it to define certain modules that will come up in our study of modular representation theory via homotopy theory.

**Definition 1.11.** For $n > 0$, we define the module $\Omega^n(M) := \ker \delta_{n-1} = \text{im} \delta_n$.

We can also dually define injective modules and injective resolutions $M \hookrightarrow I_\ast$.

**Definition 1.12.** For $n > 0$, we define the module $\Omega^{-n}(M) := \text{coker} \delta^{n-1} = \text{coim} \delta^n$.

**Definition 1.13.** For $n = 0$, we define $\Omega^0(M) := \Omega(\Omega^{-1}(M))$. This is so that $M \cong \Omega^0(M) \oplus \text{(projective)}$.

One can verify that this is well defined up to isomorphism. Furthermore, one can check that in fact, $\Omega^n(-)$ is a functor.

What if we use an arbitrary projective resolution instead of a minimal projective resolution? It turns out that taking the kernels of the differentials in our arbitrary projective resolution gives a module $\Omega^n(M)$ that differs from $\Omega^n(M)$ up to a projective module.

**Proposition 1.14.** $\widehat{\Omega}^n(M) \cong \Omega^n(M) \oplus \text{(projective)}$.

**Proposition 1.15.** These modules $\Omega^n(M)$ satisfy the following nice properties:

(i) $\Omega^n(\text{projective}) = 0$.
(ii) $\Omega^n(M \oplus N) \cong \Omega^n(M) \oplus \Omega^n(N)$.
(iii) $\Omega^n(M) \ast \cong \Omega^{-n}(M) \ast$.
(iv) $\Omega^n(\Omega^n(M)) \cong \Omega^{n+m}(M)$.
(v) $\Omega^n(M) \otimes \Omega^n(N) \cong \Omega^{n+m}(M \otimes N) \oplus \text{(projective)}$.

### 1.2. What is the Stable Module Category?

Note that the functor $\Omega^n(-)$ is almost well behaved with respect to the tensor product of $kG$-modules. What would happen if we force it to be well behaved by modding out the projective modules in $kG$-Mod? This leads us to the Stable Module Category, and it turns out to be a natural setting to do cohomology theory!

**Definition 1.16.** Given a ring $R$, the stable module category $\text{StMod}(R)$ has objects $R$-modules, and has a vector space of morphisms $\text{Hom}_R(M, N) = \text{Hom}_R(M, N)/\text{PHom}_R(M, N)$, where $\text{PHom}_R(M, N)$ is the linear subspace of projective maps (maps that factor through a projective module).
Note that $\Omega^n(-)$ is now well behaved with respect to tensor product in $\text{StMod}(R)$. Furthermore, we now also have that $\Omega$ and $\Omega^{-1}$ are now inverses.

**Theorem 1.17.** The morphisms in $\text{StMod}(kG)$ are precisely the $\text{Ext}_{kG}$ groups. That is,

$$\text{Hom}_{kG}(\Omega^n(M), N) \cong \text{Ext}^n_{kG}(M, N) \cong \text{Hom}_{kG}(M, \Omega^{-n}(N))$$

**Proof.** Sketch. Show an element of Ext is represented by a map $\Omega^n \to M$. Two maps represent the same class if they factor through a projective module. \flushright{□}

**Remark 1.18.** Note that this theorem suggests a natural definition for negative cohomology groups. This recovers Tate cohomology! Furthermore, we can do homotopy theory in this setting!

## 2. Triangulated Categories and Model Categories

$kG\text{-Mod}$ is an abelian category, but $\text{StMod}(kG)$ is only a triangulated category (kernels + cokernels don’t usually exist) with suspension given by $\Omega^{-1}$. The triangle

$$A \to B \to C \to \Omega^{-1}A$$

is distinguished iff we have short exact sequences

$$0 \to A \to B \oplus \text{proj} \to C \to 0$$

$$0 \to B \to C \oplus \text{proj} \to \Omega^{-1}A \to 0$$

with the appropriate maps equal in the stable category.

Every time we see a triangulated category, we should ask: what is the underlying homotopy category structure? In other words, is there a (stable) model category structure on $kG\text{-Mod}$? The answer is yes!

**Theorem 2.1.** The stable model structure on $kG\text{-Mod}$ has weak equivalences being the stable equivalences, the fibrations are surjections, and the cofibrations are injections.

**Definition 2.2.** We say a map $f : M \to N$ is a stable equivalence if there exists a map $g : N \to M$ such that $fg - \text{Id}, gf - \text{Id}$ factor through a projective module. (That is, $f$ has an inverse in the stable module category).

However, we can do even better than verifying that these satisfy the axioms of a model category: we can use a recognition theorem to show that $kG\text{-Mod}$ is a cofibrantly generated model category. This is great because this gives us functorial factorization! This is sketched in Hovey’s Model Categories.

**Theorem 2.3.** $kG\text{-Mod}$ is a cofibrantly generated model category, and we take $I = \{a \to kG \mid a \text{ a left ideal in } kG\}$, the inclusion of the zero ideal.

We choose $J = \{0 \to kG\}$, the inclusion of the zero ideal.

Essential to the construction of this cofibrantly generated model category is the fact that $kG$ is a Frobenius ring. In fact, Hovey makes the following remark:

**Remark 2.4** (Hovey). If there is cofibrantly generated model category structure with $I$ and $J$ the generating cofibrations and generating acyclic cofibrations, then $R$ must be Frobenius ring.

**Definition 2.5.** A ring $R$ is a Frobenius ring if the classes of projective and injective modules coincide.

**Proposition 2.6.** $kG$ is a Frobenius ring.

**Proof.** If $H \leq G$, then a $kG$-module $M$ is projective iff $M$ considered as a $kH$-module is projective. The forward direction is obvious using the direct summand of free definition, the converse involves constructing a module homomorphism for each module surjection. Also, a $kG$-module $M$ is injective iff $M$ considered as a $kH$-module is injective. The forward direction follows from the submodule/internal direct sum definition, and the converse direction is through induction (tensoring with $kG$ over $kH$).

This reduces to the case that $G$ is a $p$-group. In this case, one can show that injective modules are free. Hence a projective is a direct summand of an injective module, hence injective. \flushright{□}
**Remark 2.7.** This is a model category where every object is bifibrant. \(0 \to M \text{ and } M \to 0\) are clearly cofibrant/fibrant by our characterizations).

**Remark 2.8.** One can explain the weirdness in the definition of the stable equivalences through the theory of cotorsion pairs. A cotorsion pair on an abelian category gives rise to an abelian model structure. However, unlike most model structures, it is easy to determine what the fibrations and cofibrations and factorizations should be, and harder to define what the weak equivalences are.

A cotorsion pair \((\mathcal{D}, \mathcal{E})\) in an Abelian category \(\mathcal{A}\) is a pair of classes of objects in \(\mathcal{A}\) which are orthogonally complementary to each other with respect to the Ext functor. For a Frobenius ring \(R\), take \(\mathcal{C} = \mathcal{F}\) to be the class of all \(R\) modules and \(\mathcal{W}\) to be the class of all projective (= injective) modules.

### 3. Ideas and Machinery

One really cool thing is that we can borrow ideas and machinery from stable homotopy theory, and import them to \(\text{StMod}(kG)\). Furthermore, we can study the analogous conjectures in this setting and hope to gain insight into the setting of the stable homotopy category.

Here is a list of some theorems, buzzwords, and conjectures and their analogs:

1. **The Nilpotence Theorem:** There are periodic phenomena in stable homotopy theory (image of \(J\)). For Bott periodicity, Adams found a geometric manifestation of this by producing for each prime \(p\) a self map \(\Sigma^{k_p}M_p \to M_p\) of the mod \(p\) Moore spectrum. \((p = 2, k_p = 8,\) otherwise \(k_p = 2p - 2)\). Later (Smith, Toda) replace \(k\)-theory with complex bordism, found more self-maps of finite complexes inducing non-nilpotent endomorphisms. Ravenel conjectured that the generalised homology theory of complex bordism, \(MU_*(-)\), detects whether a map \(f : X \to Y\) between finite spectra is nilpotent. (if and only if \(MU_*(f)\) is nilpotent). Proved by Devinatz, Hopkins, Smith.

2. **The Classification of Thick Subcategories:** This was a major breakthrough by Devinatz, Hopkins and Smith in stable homotopy theory. Complex bordism is really hard to calculate, so if you \(p\)-localize the stable category, and you get Morava K-theories. These are easier to compute with. \(K(0)\) is \(HQ\), \(K(\infty) = HZ/p\). Can also detect nilpotence.

   Let \(\mathcal{F}\) denote the category of compact objects in the \(p\)-local stable homotopy category of spectra. If \(R\) is a non-trivial ring spectrum, then there exists an \(n, (0 \leq n \leq \) ) such that \(K(n)_*R \neq 0\).

   Let \(\mathcal{C}_n := \{X \in \mathcal{F} : K(n - 1)_*X = 0\}\). Thick subcategories (full, triangulated, closed under extensions) are precisely the \(\mathcal{C}_n\).

   Suppose \(P\) is some generic property of spectra and we want to identify the subcategory of finite spectra which satisfy \(P\). If we can find a type-\(k\) spectrum which satisfies \(P\) and a type-\((k1)\) spectrum which does not satisfy \(P\), that forces the subcategory in question to be \(\mathcal{C}_k\).

   Thick tensor-ideal subcategory of \(\text{StMod}(kG)\) must be of the form \(C(X)\) for some nonempty set \(X\) of closed homogeneous subvarieties (which is closed under specialization and finite unions).

   The analog for this in the stable module category are the kappa modules of Benson and Greenlees.

3. **The Generating Hypothesis:** (Freyd) If \(\Phi : X \to Y\) is a map between finite spectra such that \(\pi_*(\Phi) = 0\), then \(\Phi\) is nullhomotopic. \(\pi_*\) from finite spectra to \(\pi_*(S^0)\)-modules is faithful). No finite spectrum other than wedge of suspensions of sphere spectra whose homotopy groups are finitely generated over \(\pi_*(S^0)\). Therefore, the GH, if true, reduces the study of finite spectra \(X\) to the study of their homotopy groups \(\pi_*(X)\) as modules over \(\pi_*(S^0)!\)

   Benson, Chebolu, Christensen, Minac formulate analogous statements for \(\text{StMod}(kG)\). \(\Phi : M \to N\) between finite dimensional \(kG\)-modules is trivial in \(\text{stmod}(kG)\) if the induced map in Tate cohomology \(\text{Hom}(\Omega^i k, M) \to \text{Hom}(\Omega^i k, N)\) is trivial for each \(i\). Holds iff \(G = \mathbb{Z}/2\) or \(G = \mathbb{Z}/3\). Note does not depend on \(k\), so long as characteristic is correct.

### 4. Phantom Maps + Purity

In algebraic topology, we have various definitions (unstable and stable) of the notion of a phantom map:

**Definition 4.1.** A map \(f : X \to Y\) between CW complexes is phantom if its restriction to each skeleton \(X^{(n)}\) is nullhomotopic. A map \(f : X \to Y\) between topological spaces is phantom if the composition with any map from finite dimensional complexes is nullhomotopic.
Lemma 4.2. The composite of phantom maps is nullhomotopic.

Definition 4.3. A map $f : X \to Y$ between spectra is phantom if the composition with any map from finite spectra is nullhomotopic.

Lemma 4.4. Equivalently, the corresponding map on homology theories $h_f : h_X \to h_Y$ is zero.

We will prove that phantom maps exist, but first, what does their existence tell us? Looking at (co)homology is not always enough: this is perhaps one motivation for looking at spectra instead!

Theorem 4.5 (Gray). There exist uncountably many phantom maps from $\mathbb{CP}^\infty \to S^3$.

It’s really crappy/hard to construct an explicit phantom map. It’s done in More Concise Algebraic Topology, also in Gray’s paper.

In contrast, in modular representation theory there can never be a phantom map whose target is a finite-dimensional module.

Christensen and Strickland provide categorical framework to prove properties about phantom maps. In a Brown representable category (see below), the composite of two phantom maps is always trivial. However, not every triangulated category is Brown representable. For example, consider the derived category of unbounded complexes of modules over a polynomial ring in two variables over a field $k$. This category is Brown representable iff $k$ is countable.

We recall Neeman’s result on Brown representability for triangulated categories.

Theorem 4.6 (Neeman). Let $\mathcal{T}$ be a compactly generated triangulated category. Let $H : \mathcal{T}^{op} \to \text{Ab}$ be a homological functor. That is, $H$ is contravariant and takes triangles to long exact sequences. Suppose the natural map

$$H(\bigoplus_{\lambda \in \Lambda} t_\lambda) = \prod_{\lambda \in \Lambda} H(t_\lambda)$$

is an isomorphism for all small coproducts in $\mathcal{T}$. Then $H$ is representable.

Definition 4.7. A homology theory on a triangulated category $\mathcal{T}$ is an exact covariant functor to an abelian category that takes direct sums in $\mathcal{T}$ to direct sums.

Remark 4.8. We take our abelian category to be $\text{Vect}(k)$.

Definition 4.9. We say that a category $(\text{StMod}(kG))$ is Brown representable if all homology theories are Brown Representable, and every map between homology theories is representable. (i.e. for all homology theories $F$ there exists a $kG$-module $M$ such that $F(N) \cong \text{Hom}_{kG}(k, M \otimes_k N)$).

Theorem 4.10 (Christensen, Strickland, Neeman, Phantom Maps and Purity). The following conditions are equivalent:

(i) Brown representability holds for the category $\text{StMod}(kG)$.

(ii) The group algebra $kG$ has pure global dimension zero or one.

(iii) Either $k$ is countable or $G$ has cyclic Sylow $p$-subgroups.

REFERENCES


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