NOTES ON VECTORS, PLANES, AND LINES

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1. Vectors

I assume that the reader is familiar with the basic notion of a vector. The important feature of the vector is that it has a magnitude and direction. If v is a vector, say

$$v = (v_1, v_2, \ldots, v_n)$$

then its magnitude or **norm** is given by

$$\boxed{\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.}$$

There is a natural correspondence between vectors and points. If you have a point P in n-dimensional Euclidean space, say

$$P = (P_1, P_2, \ldots, P_n),$$

then we can associate to P, the vector v given by

$$v = (P_1, P_2, \ldots, P_n).$$

We think of the vector v as starting as the origin and ending at the point P.

Given two points P and Q, we can naturally form a vector v which starts at P and ends at Q. If

$$P = (P_1, P_2, \ldots, P_n)$$
$$Q = (Q_1, Q_2, \ldots, Q_n),$$

then the vector $\overline{PQ}$ is given by

$$\overline{PQ} = (Q_1 - P_1, Q_2 - P_2, \ldots, Q_n - P_n).$$

We add and subtract vectors in the natural way. If

$$v = (v_1, v_2, \ldots, v_n)$$
$$w = (w_1, w_2, \ldots, w_n),$$

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then we define

$$v + w = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$
$$v - w = (v_1 - w_1, v_2 - w_2, \ldots, v_n - w_n).$$

Scalar multiplication is also a natural operation. If v is a vector as above, and λ is a real number, then we define **scalar multiplication** by

$$\lambda v = (\lambda v_1, \lambda v_2, \ldots, \lambda v_n).$$

Here, we are *scaling* the magnitude of v, but not changing its direction.[1] In some cases, the direction of a vector is the feature of interest. When this is the case, we often scale are vectors as to have magnitude 1. Such a vector is called a **unit vector**. If v is as above, then we can form a unit vector u that points in the direction of v. We define u by

$$u = \frac{v}{\|v\|}.$$

We leave it as an exercise to verify that u does have norm 1. In fact, this is a specific case of a more general result. If λ is a real number and v a vector as above, then

$$\boxed{\|\lambda v\| = |\lambda| \, \|v\| \, .}$$

As a consequence of this, if v is a vector, we can find a vector $\widetilde{v}$ that points in the direction of v and has norm λ as follows:

$$\widetilde{v} = \frac{\lambda v}{\|v\|}.$$

Another important operation is the **dot product** or **inner product** of two vectors. With v and w as above, we define the dot product by

$$\boxed{v \cdot w = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.}$$

Observe that $v \cdot w$ is a real number and not a vector.

If our vectors are 3-dimensional, then we can define a special operation called the **cross product**. If

$$v = (v_1, v_2, v_3)$$
$$w = (w_1, w_2, w_3),$$

then we define the cross product by

$$\boxed{v \times w = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} .}$$

[1] Really we are not changing the 1-dimensional subspace spanned by v. If $\lambda < 0$, we flip v, or reflect v about the origin.

This idea can be generalized to other spaces of vectors, but we have no need for this here.

2. Geometry of n-dimensional Space

Now that we have defined some basic operations, we can delve into some geometric aspects.

First, we introduce a basic idea of angle between vectors. From this, one can discuss the notions of parallel and orthogonal.

We define the **angle between two vectors** v **and** w, denoted by $\angle_{v,w}$, by

$$\boxed{\cos \angle_{v,w} = \frac{v \cdot w}{\|v\| \ \|w\|}.}$$

This generalizes the idea of angle in the plane. A moments thought should reveal that we are actually measuring this angle in the plane. Specifically, we are measuring the angle between v and w in the plane spanned by v and w. You should verify for 2-dimensional vectors that this does indeed give you the angle as we know it.

From this, we say that two vectors v and w are **parallel** if and only if

$$\boxed{\cos \angle_{v,w} = \pm 1.}$$

That is, the angle between v and w is 0 or π.

Exercise 1. Show that v and w, both nonzero vectors, are parallel if and only if there exists a real number λ such that

$$\lambda v = w.$$

The above exercise gives an alternative formulation of parallel which is useful.

Likewise, we say two vectors v and w are **orthogonal** if and only if

$$\boxed{\cos \angle_{v,w} = 0.}$$

An easy consequence of the definition of $\angle_{v,w}$ is that two vectors v and w which are nonzero are orthogonal if and only if

$$v \cdot w = 0.$$

Orthogonality of vectors is useful. Observe that if we have two vectors, v and w, we can write v as a vector parallel to w and a vector orthogonal to w. For this, we introduce the idea of projection of v onto w.

The **projection of** v **onto** w is defined by

$$\boxed{\operatorname{Proj}_w v = \|v\| \cos \angle_{v,w} \frac{w}{\|w\|}.}$$

This can be simplified by the following computation:

$$\begin{aligned}
\operatorname{Proj}_w v &= \|v\| \cos \angle_{v,w} \frac{w}{\|w\|} \\
&= \|v\| \left(\frac{v \cdot w}{\|v\| \, \|w\|} \right) \frac{w}{\|w\|} \\
&= \frac{v \cdot w}{\|w\|^2} w.
\end{aligned}$$

Alternatively, we have

$$\boxed{\operatorname{Proj}_w v = \frac{v \cdot w}{\|w\|^2} w.}$$

We call the number

$$\frac{v \cdot w}{\|w\|^2}$$

the **component of** v **on** w. Lastly, let u be the vector that satisfies the equation

$$\operatorname{Proj}_w v + u = v.$$

Exercise 2. Show that u is orthogonal to w. That is

$$w \cdot u = 0.$$

Now in 3-dimensional space, if one is given two distinct vectors v and w, i.e., v and w are not parallel, then we can find a vector u that completes 3-dimensional space. That is, every vector x can be written in the form

$$\lambda_1 v + \lambda_2 w + \lambda_3 u = x.$$

Now, if we further require that u be orthogonal to v and w, then we are limited on our choices.[2] Luckily for us, we have defined an operation that will give us such a u. Namely, $v \times w$. Another way to form such a u is to take a vector u that only satisfies our first condition that v, w, and u complete 3-dimensional space. Under this condition,

$$\operatorname{Proj}_v u \neq u$$

and

$$\operatorname{Proj}_w u \neq u.$$

[2]Think about the geometric picture. There is essentially one direction that u can point if we insist on the second condition.

Then we can find vectors u_1 and u_2 that are orthogonal to v and w. Then a suitable linear combination of u_1 and u_2 can be taken as to get a new vector $\widetilde{u}$ which is orthogonal to both v and w. Then, since we can write

$$u = \alpha_1 v + \alpha_2 w + \alpha_3 \widetilde{u},$$

v, w, and $\widetilde{u}$ will complete 3-dimensional space.

Exercise 3. Verify that

(a)

$$v \cdot (v \times w) = 0.$$

(b)

$$w \cdot (v \times w) = 0.$$

We now give some important properties of the cross product. For those familiar with the determinant, it should be no great shock that we can measure areas and volumes using the cross product, since determinants measure these things.[3]

Let v and w be 3-dimensional vectors. Then the **area of the parallelogram formed by v and w**, denoted by $\mathcal{P}$, is given by

$$\boxed{A(\mathcal{P}) = \|v \times w\|.}$$

If we have three vectors v, w, and u, then the **volume of the parallelepiped formed by u, v, and w**, denotes by $\mathcal{P}$, is given by

$$\boxed{V(\mathcal{P}) = |u \cdot (v \times w)|.}$$

3. Lines

As everyone knows, when walking on a line, there are two directions one can move, forwards and back. Viewing these directions as one being the negative of the other, you have one degree of freedom on a line. In 2-space, one equation in the variables x and y of the form

$$ax + by = c,$$

determines a line. Solving for y, we get

$$y = mx + p,$$

where m is the slope and p the y-intercept. Thinking of p as just a special point of the line and m as a direction, we see that a line is determined by a point and a direction. Notice that x is the degree

[3] Really, eigenvalues measure these things. But the determinant of a matrix is the product of its eigenvalues. For those familiar with eigenvalues, recall the eigenvalue measures contraction and expansion in the coordinate directions.

of freedom, since y is completely determined once we have selected a particular x.

In higher dimensions, we utilize vectors. If P and Q are points in n-space, then we can form the line connecting P and Q by extending the line segment connecting P and Q. As we saw earlier, there is a natural vector associated to P and Q, and to P. Then the equation of a line in n-space is the vector valued equation

$$\boxed{r(t) = \overline{PQ}t + \overline{P},}$$

where $\overline{PQ}$ is the vector that starts at P and ends at Q, and $\overline{P}$ is the vector that starts at the origin and ends at P. The variable t is our one degree of freedom. If we expand this in each of the components, we actually get $n-1$ equations. From this, we see that 1-dimensional objects need $n-1$ distinct equations. In general j-dimensional objects need $n-j$ equations. To summarize,

$$\boxed{\text{A Line is determined by a point and a vector.}}$$

In 2-space, the Parallel Postulate asserts that two lines either intersect once or are parallel. In n-space, $n > 2$, this is not the case. Two lines need not intersect. However, if two lines intersect, we can define the angle of intersection. Let ℓ_1 and ℓ_2 be two lines in n-space which intersect. Now, from above, we know that

$$\begin{aligned} \ell_1 : \quad & r_1(t) = d_1 t + p_1 \\ \ell_2 : \quad & r_2(t) = d_2 t + p_2. \end{aligned}$$

Then we define the angle between ℓ_1 and ℓ_2, denotes by $\angle_{\ell_1,\ell_2}$, by

$$\boxed{\cos \angle_{\ell_1,\ell_2} = \frac{|d_1 \cdot d_2|}{\|d_1\| \ \|d_2\|}.}$$

Observe that this is just the angle $\angle_{d_1,d_2}$ between the two direction vectors.

Once we defined the notion of angle, we can again define the notions of parallel and orthogonal.

Two lines ℓ_1 and ℓ_2 are **parallel** if and only if the direction vectors for the two lines are parallel. That is

$$\boxed{\angle_{\ell_1,\ell_2} = \angle_{d_1,d_2} = \pi \text{ or } 0.}$$

Two lines ℓ_1 and ℓ_2 are **orthogonal** if and only if the direction vectors for the lines are orthogonal. That is

$$\boxed{\angle_{\ell_1,\ell_2} = \angle_{d_1,d_2} = \frac{\pi}{2}.}$$

4. Planes

We treat planes only in 3-space for simplicity. Observe that from a comment above, in n-space, a plane which is a 2-dimensional object would need $n - 2$ distinct equations to describe. Then, since

$$3 - 2 = 1,$$

we see in 3-space, planes should only need 1 equation.

Now, assuming that you cannot jump up and down, and since the Earth locally looks like a plane, we know when we walk, we have basically four possible directions. North, South, East, and West. Since North and South are negatives, and East and West are negatives, we see that we have 2 degrees of freedom, or two directions.

Given three distinct points, not sitting on the same line (not colinear), then we can form a plane. Now, we know that from three points, we can form two vectors (North and East). Then every direction you can move in the plane can be described by moving in these two directions. That is, a point lines in the plane if and only if the vector you form from this point and any of our first three points should be expressible as a linear combination of our two direction vectors (North and East). This gives us one equation. Now, there is an alternative. Let v and w be our direction vectors forms from the points P, Q, and R. We saw earlier that there is a unique direction which is orthogonal to both v and w, and completes 3-space. This vector is given by $v \times w$. Now, we take any point in the plane, say S, and form the form $\overline{PS}$, then $\overline{PS}$ is expressible by

$$\overline{PS} = \lambda_1 v + \lambda_2 w.$$

So that

$$\overline{PS} \cdot v \times u = 0.$$

This leads us to the following equation:

$$\boxed{(u \times v) \cdot (x - P_1, y - P_2, z - P_3) = 0.}$$

We call $u \times v$ the **normal vector** to the plane, and write this as just n. Then the above becomes

$$\boxed{n \cdot (x - P_1, y - P_2, z - P_3) = 0.}$$

Expanding this out, we could solve for x, y, or z, in terms of the other two. Thus, we have two degrees of freedom. To summarize:

$$\boxed{\text{A Plane is determined by a normal vector and a point,}}$$

and

$$\boxed{\text{The normal vector is determined by two vectors in the plane.}}$$

Now, for planes, something along the lines of the Parallel Postulate is true. That is, two planes $\mathcal{P}_1$ and $\mathcal{P}_2$ are either parallel or intersect in a line.

From this we define the angle between two planes, $\mathcal{P}_1$ and $\mathcal{P}_2$ of the form

$$\begin{aligned} \mathcal{P}_1 : &\quad n_1 \cdot (x - P_1, y - P_2, z - P_3) = 0 \\ \mathcal{P}_2 : &\quad n_2 \cdot (x - Q_1, y - Q_2, z - Q_3) = 0, \end{aligned}$$

by

$$\boxed{\angle_{\mathcal{P}_1,\mathcal{P}_2} = \frac{|n_1 \cdot n_2|}{\|n_1\| \ \|n_2\|}.}$$

Observe that this is just

$$\boxed{\angle_{\mathcal{P}_1,\mathcal{P}_2} = \angle_{n_1,n_2}.}$$

From this we can again define the notion of parallel and orthogonal.

Two planes $\mathcal{P}_1$ and $\mathcal{P}_2$ are **parallel** if and only if

$$\boxed{\angle_{\mathcal{P}_1,\mathcal{P}_2} = 0 \text{ or } \pi.}$$

Two planes $\mathcal{P}_1$ and $\mathcal{P}_2$ are **orthogonal** if and only if

$$\boxed{\angle_{\mathcal{P}_1,\mathcal{P}_2} = 0.}$$

Now that we have defined the idea of lines and planes in 3-space, we might wonder about the interplay between planes and lines. The first thing we must introduce is the angle between a plane and a line. Observe that if we have a line ℓ and plane $\mathcal{P}$, then they either intersect at a point or are parallel.

Let ℓ be a line of the form

$$\ell : \quad r(t) = dt + p,$$

and $\mathcal{P}$ be a plane of the form

$$\mathcal{P} : \quad n \cdot (x - P_1, y - P_2, z - P_3).$$

We define the **angle between ℓ and $\mathcal{P}$**, denoted by $\angle_{\ell,\mathcal{P}}$ by

$$\boxed{\angle_{\ell,\mathcal{P}} = \frac{\pi}{2} - \cos^{-1}\left(\left|\frac{n \cdot d}{\|n\| \ \|d\|}\right|\right).}$$

This is a rather complicated thing. A moments thought will tell you it must be.

From this we are lead to the following table:

Geometric Objects	**Direction Vectors**
$\ell_1 \parallel \ell_2$	$d_1 \parallel d_2$
$\ell_1 \perp \ell_2$	$d_1 \perp d_2$
$\mathcal{P}_1 \parallel \mathcal{P}_2$	$n_1 \parallel n_2$
$\mathcal{P}_1 \perp \mathcal{P}_2$	$n_1 \perp n_2$
$\mathcal{P} \parallel \ell$	$n \perp d$
$\mathcal{P} \perp \ell$	$n \parallel d$

Observe the reversal in the condition that a plane be parallel to a line and a plane be orthogonal to a line. This is because of how we define the plane, via the direction that is orthogonal to the plane. One can verify the validity of this with a few simple drawings.

5. Getting Geometric Objects From Geometric Constraints

One of the main problems that arise are problems where we are to find a geometric object like a line or plane based on geometric conditions. Some conditions are to require that the line or plane be parallel or orthogonal to a known plane or line. Or that the plane or line contain a certain set of points. The above table can be used to obtain the vector conditions needed to obtain a vector and a point.

Given some bit of geometric information, the idea is to translate this into the language of vectors. We do this because the equations for lines and planes require vectors. **Always Remember** that a line is determined by a direction and a point and a plane is determined by a normal vector and a point. When reading problems, on cannot lose sight of this goal. Always ask the question, how can I get the direction or normal vector from this information. When lost, draw a generic picture of what is going on. Often you are lead to the conditions of the above table.

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