Contemporary Mathematics

Banach and operator space structure of C^*-algebras

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Introduction

A C^*-algebra is often thought of as the non-commutative generalization of a $C(K)$-space, i.e. the space of continuous functions on some locally compact Hausdorff space, vanishing at infinity. We go one step further, for we seek to compare the Banach space properties of C^*-algebras *and* their naturally complemented subspaces, with those of $C(K)$-spaces. (For a recent survey on $C(K)$ spaces or Banach spaces, see [**Ro3**].) This involves the recent theory of operator spaces, or quantized Banach spaces. We briefly review this concept at the beginning of section 1; the reader is referred to [**ER**] and [**Pi**] for in depth coverage. For the definition of complete boundedness of maps, complete isomorphisms, etc., see section 1.

Our presentation here is expository; only simple deductions are given, often from rather deep principles.

Section 1 shows how C^*-algebras share certain Banach space properties of $C(K)$-spaces. For example, we state Pfitzner's theorem that C^*-algebras have Pełczyński's property (V) as Theorem 1.1, and then deduce that non-reflexive completely complemented subspaces of C^*-algebras contain complete isomorphic copies of c_0 in Corollary 1.2. We then use an old result

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of the author's to deduce that non-reflexive completely complemented subspaces of von Neumann algebras contain complete isomorphic copies of ℓ^∞, in Corollary 1.3.

We next discuss the important class of *nuclear* C^*-algebras; in many ways, these are the closest, in Banach space structure, to $C(K)$-spaces. These include C^*-algebras with separable duals, and more generally, type I C^*-algebras. These are described via Definition 1.5, after which we give a description of the CAR, or Fermion algebra, a fundamental nuclear non-type I (separable) C^*-algebra. Section one concludes with remarkable results of Glimm and Kirchberg. In particular, Kirchberg proved the non-commutative analogue of Milutin's theorem: *Every non-type I nuclear separable C^*-algebra is completely isomorphic to the CAR algebra*.

Section two deals with quantized versions of the separable extension property (SEP) for Banach spaces. These were introduced in [**Ro2**] and developed further in [**OR**] and [**AR**]. Recall that a separable Banach space has the SEP provided it is complemented in every separable superspace. Sobczyk established that c_0 *has* the SEP, and Zippin proved that c_0 is the *only* separable space with this property, up to isomorphism. We give a proof of Sobczyk's theorem following Theorem 2.7, which motivates the approach taken in [**AR**] (and was also given there). This proof uses the Borsuk extension theorem (Theorem 2.7).

Our first quantized result of the SEP: *A separable operator space has the CSEP provided it is completely complemented in every separable operator superspace.* (This is equivalent to the definition given in Section 2, Definition 2.2, in virtue of the injectivity of $B(H)$, Theorem 2.3). The spaces ROW and COLUMN, denoted R and C, are defined following the above mentioned proof, and then the author's discovery from [**Ro2**] (as refined in [**AR**]) is given as Theorem 2.12: $c_0(M_{n,\infty} \oplus M_{\infty,n})$ *has the 2-CSEP for all* n. Of course this implies that $c_0(R \oplus C)$ has the CSEP. There follows some discussion of a possible converse to this, namely the conjecture: a separable operator space with the CSEP completely embeds in $c_0(R \oplus C)$. This conjecture is at a far deeper level than the results obtained so far.

Unfortunately, $\mathbf{K}$ (the space of all compact operators) *fails* the CSEP, as discovered by Kirchberg. We give some discussion of an alternate proof, from [**OR**], showing that there is a separable operator superspace Y of K_0 with Y/K_0 completely isometric to c_0 and K_0 completely uncomplemented in Y (Theorem 2.9). (K_0 denotes the c_0-sum of M_n's.) T. Oikhberg and the author succeeded in proving that $\mathbf{K}$ has our second quantized version of the SEP, the CSCP (Theorem 2.12). In virtue of a discovery in [**OR**], this property may be formulated: *a locally reflexive separable operator space has the CSCP provided it is completely complemented in every locally reflexive separable operator superspace.* This suggests the obviously quite deep problem: if an operator has the CSCP, does it completely embed in $\mathbf{K}$?

The last part of Section 2 deals with the approach taken in [**AR**], namely that of fundamental properties of complete M-ideals (see Definition 2.15).

We do not discuss here the basic new tool used in [**AR**] for dealing with these, *M-complete approximate identities.* Rather, we just show the basic principles developed in [**AR**] which lead to Theorems 2.8 and 2.12. Among these are the lifting result from [**AR**], stated as Theorem 2.21 here, which says qualitatively that *if $\mathcal{J} \subset Y \subset \mathcal{A}$ with $\mathcal{J}$ a nuclear ideal in a C^*-algebra and Y locally reflexive with $Y/\mathcal{J}$ separable, then $\mathcal{J}$ is completely complemented in Y.*

The proof of Theorem 2.21 in [**AR**] uses ideas from work of Effros-Haagerup [**ER**], where this result is established for $Y = \mathcal{A}$ itself. An alternative proof of a pure operator space isometric extension of the latter is given in the appendix to [**AR**], and formulated here as Theorem 2.16. The article concludes with a sketch of the proof that $\mathbf{K}$ has the CSCP. Several open problems are also discussed, in both Sections 1 and 2.

1. C^*-algebras from a Banach space perspective

What do C^*-algebras look like, as Banach spaces, up to linear homeomorphism? What do their naturally complemented subspaces look like? The appropriate way to approach these questions is through the theory of *quantized* Banach spaces, or operator spaces, whose natural morphisms are completely bounded maps. We first briefly recall this concept; for fundamental background and references, see [**ER**] and [**Pi**].

An *operator space* X is a complex Banach space which is a closed linear subspace of $B(H)$, the bounded linear operators on some Hilbert space H, endowed with its natural tensor product structure with $\mathbf{K} = K(\ell_2)$ (where $K(H)$ denotes the space of compact operators on H). We let $\mathbf{K} \otimes_{\mathrm{op}} X$ denote the closed linear span in $B(\ell_2 \otimes_2 H)$ of the operators $A \otimes T$ where $A \in \mathbf{K}$, $T \in X$, and $\ell_2 \otimes_2 H$ is the Hilbert space tensor product of ℓ_2 and H. A linear operator $T : X \to Y$ between operator spaces X and Y is called *completely bounded* if $I_{\mathbf{K}} \otimes T$ is a bounded linear operator from $\mathbf{K} \otimes X$ to $\mathbf{K} \otimes Y$, endowed with their natural norms. Of course, $I_{\mathbf{K}} \otimes T$ then uniquely extends to a bounded linear operator from $\mathbf{K} \otimes_{\mathrm{op}} X$ to $\mathbf{K} \otimes_{\mathrm{op}} Y$, which we also denote by $I_{\mathbf{K}} \otimes T$; then we *define* $\|T\|_{\mathrm{cb}} = \|I_{\mathbf{K}} \otimes T\|$. It now follows easily that if X_i, Y_i are operator spaces and $T_i : X_i \to Y_i$ are completely bounded maps, then also $T_1 \otimes T_2$ is (i.e., extends to) a completely bounded map from $X_1 \otimes_{\mathrm{op}} X_2$ to $Y_1 \otimes_{\mathrm{op}} Y_2$, with $\|T_1 \otimes T_2\|_{\mathrm{cb}} \le \|T_1\|_{\mathrm{cb}} \|T_2\|_{\mathrm{cb}}$. We should point out that when X_1 and X_2 are C^*-algebras, $X_1 \otimes_{\mathrm{op}} X_2$ is also called the *spatial tensor product*, and also the *minimal* tensor product, for it is the least tensor norm on $X_1 \otimes X_2$ whose completion (i.e. $X_1 \otimes_{\mathrm{op}} X_2$) is a C^*-algebra (cf. [**Mu**] for a proof of this theorem).

Now many natural Banach space concepts naturally extend to the context of operator spaces. Thus operator spaces X and Y are called *completely isomorphic* if there exists an invertible linear operator $T : X \to Y$ with T and T^{-1} completely bounded. If $\|T\|_{\mathrm{cb}} \|T^{-1}\|_{\mathrm{cb}} \le \lambda$, we say X and Y are λ-*completely isomorphic*; then we set $d_{\mathrm{cb}}(X, Y) = \inf\{\lambda \ge 1 :$

X is λ-completely isomorphic to $Y\}$. If $X \subset Y$ with Y an operator space and X a closed linear subspace, then X is regarded as an operator space via the natural structure $\mathbf{K} \otimes_{\rm op} X \subset \mathbf{K} \otimes_{\rm op} Y$. X is then called *completely complemented in* Y if there is completely bounded (linear) projection P mapping Y onto X. If $\|P\|_{\rm cb} \le \lambda$, we say X is λ-*completely complemented in* Y; if $\|I - P\|_{\rm cb} \le \lambda$, we say X is λ-*completely co-complemented in* Y.

Now of course $\mathbf{K}$ may be identified with those infinite matrices representing compact operators on ℓ_2 with respect to its natural basis. For an operator space X, $\mathbf{K} \otimes_{\rm op} X$ may also be identified with a Banach space of infinite matrices with elements in X. Now let M_{00} denote all infinite matrices of scalars with only finitely many non-zero entries. Then if M_n denotes the space of $n \times n$ matrices of complex numbers, we may regard $M_0 \subset M_1 \subset \cdots \subset M_n \subset M_{n+1} \subset \cdots \subset M_{00} \subset \mathbf{K} = \overline{M}_{00}$. It follows easily that if $P_n : \mathbf{K} \to M_n$ in the canonical projection, then

$$P_n \otimes I_X \to I_{\mathbf{K}} \otimes I_X \quad \text{in the SOT on } K \otimes_{\rm op} X \tag{1.1}$$

(SOT denotes the Strong Operator Topology). For operator spaces X and Y and $T : X \to Y$ a bounded linear operator, we define $\|T\|_n$ by

$$\|T\|_n = \|P_n \otimes T\| \ . \tag{1.2}$$

(Equivalently, $\|T\|_n = \|I_n \otimes T\|$, where I_n denotes the identity operator on $B(\ell_2) = M_n$). It then follows easily that T is completely bounded iff $(\|T\|_n)_{n=1}^\infty$ is bounded, and then

$$\|T\|_{\rm cb} = \sup_n \|T\|_n \ . \tag{1.3}$$

(This easy equivalence is often taken as the definition of complete boundedness, c.f. [**ER**]). Identifying $\mathbf{K} \otimes_{\rm op} X$ with infinite matrices, we then have that a bounded linear operator $T : X \to Y$ is completely bounded when (Tx_{ij}) belongs to $\mathbf{K} \otimes_{\rm op} Y$ for all (x_{ij}) in $\mathbf{K} \otimes_{\rm op} X$, and then $(I_{\mathbf{K}} \otimes T)(x_{ij}) = (Tx_{ij})$.

Evidently, the concept of an operator space X is captured by the Banach space $K \otimes_{\rm op} X$. Remarkable axioms of Z. J. Ruan abstractly characterize this tensor product, without reference to the ambient Hilbert space. For this and the “correct” notion of duality, as well as various other tensor products on operator spaces, see [**ER**], [**Pi**], and also [**BP**] for the latter.

Next we recall some basic concepts concerning C^*-algebras (see also standard references such as [**Mu**] and [**Ped**]). A (concrete) C^*-algebra $\mathcal{A}$ is defined to be a norm closed subalgebra of $B(H)$ (for some H) so that $T^* \in \mathcal{A}$ for all $T \in \mathcal{A}$. Ruan's axioms have their deep historical precedent in the abstract axioms given by Gelfand and Naimark, and we make no real distinction between abstract and concrete C^*-algebras. A von-Neumann algebra $\mathcal{N}$ is defined to be a unital adjoint-closed subalgebra of $B(H)$ which is *closed* in the SOT. Since $B(H)$ is naturally a dual space, with predual C_1 (the space of trace class operators on H), it follows that von Neumann algebras are also dual spaces, for they are weak* closed in $B(H)$. A remarkable result of Sakai asserts that conversely, any C^*-algebra $\mathcal{A}$ which is isometric to a dual space (just as a Banach space) is $*$-isomorphic to a von Neumann algebra, and moreover if X and Y are Banach spaces with X^* and Y^* isometric to $\mathcal{A}$, then X and Y are isometric; thus von Neumann algebras have unique preduals, up to isometry.

Standard results show that C^*-algebras have a unique operator space structure. In fact, if $\mathcal{A}_1, \mathcal{A}_2$ are C^*-algebras and $T : \mathcal{A}_1 \to \mathcal{A}_2$ is just an algebraic surjective $*$-isomorphism, then T is already a complete isometry. However every infinite dimensional Banach space X has many possible operator space structures. Two of these are distinguished as being the smallest, denoted MIN, and the largest, denoted MAX. These are functorially presented as follows; for any operator spaces X and Y and bounded linear map $T : Y \to (X, \mathrm{MIN})$, T is completely bounded and $\|T\| = \|T\|_{\mathrm{cb}}$; for any bounded linear map $T : (X, \mathrm{MAX}) \to Y$, T is completely bounded and $\|T\| = \|T\|_{\mathrm{cb}}$.

We now deal with some fundamental Banach space results for C^*-algebras. The first one is due to H. Pfitzner [**Pf**] (c_0 denotes the space of scalar sequences tending to 0; ℓ^∞ the space of all bounded sequences of scalars).

Theorem 1.1. *Let $\mathcal{A}$ be a C^*-algebra, X a Banach space and $T : \mathcal{A} \to X$ a non weakly compact operator. There exists a commutative C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ and a subspace Y of $\mathcal{B}$ isometric to c_0 so that $T|Y$ is an isomorphism.*

Remark. The commutative version of this result (i.e. when $\mathcal{A}$ itself is commutative) is due to A. Pełczyński [**Pe2**].

Corollary 1.2. *A non-reflexive completely complemented subspace of a C^*-algebra contains a subspace completely isomorphic to c_0.*

Proof. Let $P : \mathcal{A} \to X$ be a completely bounded projection onto X, with $\mathcal{A}$ a C^*-algebra, X a non-reflexive subspace. Then of course, P is non-weakly compact. Now choose $\mathcal{B}$ and Y as in 1.1, it follows that $\mathcal{B}$ must be endowed with the MIN operator structure. Since $P|Y$ is completely bounded, $P(Y)$ must *also* have an operator space structure equivalent to MIN, whence $P|Y$ is a complete isomorphism and so $P(Y)$ is completely isomorphic to c_0. $\square$

For the next result, recall that a Banach space is isometrically injective if it is contractively complemented in every superspace. Equivalent formulations and the operator space version will be given in the next section. For any measure μ on a measurable space, $(L1(\mu))^*$ is isometrically injective, and in fact every commutative von Neumann algebra is isometric to such a space (and μ and the measurable space may be chosen with $(L1(\mu))^* = L^\infty(\mu)$). We now apply this, Theorem 1.1, and a result of the author's to obtain the von-Neumann algebra version of the previous result.

Corollary 1.3. *A non-reflexive completely complemented subspace of a von Neumann algebra contains a subspace completely isomorphic to ℓ^∞.*

Proof. Let P, $\mathcal{A}$, X, $\mathcal{B}$ and Y be as in the proof of 1.2, with $\mathcal{A}$ a von Neumann algebra. Let then $\mathcal{N} = \mathcal{B}''$, the double commutant of $\mathcal{B}$, which by a standard theorem due to von Neumann, equals the von Neumann algebra generated by $\mathcal{B}$. $\mathcal{N}$ is commutative, and hence is isometric to $(L1(\mu))^*$ for some $L1(\mu)$-space. So $\mathcal{N}$ is isometrically injective. Since $P|\mathcal{N}$ is not weakly compact, it follows by a result of the author's [**Ro1**] that there exists a subspace Z of $\mathcal{N}$ with Z isomorphic to ℓ^∞ and $P|Z$ an isomorphism. Again, since $\mathcal{N}$ has MIN as its operator space structure, $P|Z$ is a complete isomorphism and so $P(Z)$ is completely isomorphic to ℓ^∞. □

Remarks. 1. Corollaries 1.2 and 1.3 are obtained in [**Ro2**] as Proposition 2.19, with a somewhat different argument for 1.3. As noted there, it follows also that 1.2 and 1.3 also hold if one just deletes the term "completely" in their statements.

2. Unlike the commutative case, a complemented reflexive subspace of a C^*-algebra may be infinite dimensional. It is a result of G. Pisier that any such space must be isomorphic to a Hilbert space, (c.f. [**R**], Theorem 13). This suggests the following problem in the operator space category: *Characterize the Hilbertian operator spaces which are completely isomorphic to completely complemented subspaces of C^*-algebras.*

The classical Gelfand-Naimark Theorem asserts that every commutative C^*-algebra is a $C(K)$ space, i.e. $*$-isomorphic to the space of continuous functions vanishing at infinity on a locally compact Hausdorff space K. However, in terms of their Banach space structure, many C^*-algebras seem far removed from $C(K)$ spaces; for example, if H is infinite dimensional, $B(H)$ fails the approximation property [**Sz**], but $C(K)$ spaces he the metric approximation property in a natural way. From this perspective, the following class of C^*-algebras might be viewed as the "correct" non-commutative version of $C(K)$ spaces.

Definition 1.4. *A C^*-algebra $\mathcal{A}$ is nuclear provided for every finite dimensional subspace F of $\mathcal{A}$ and $\varepsilon > 0$, there exists a finite rank $T : \mathcal{A} \to A$ so that $\|Tf - f\| \le \varepsilon\|f\|$ for all $f \in F$, such that there exist n and compete contractions $U : \mathcal{A} \to M_n$ and $V : M_n \to A$ with $T = VU$.*

We have taken the operator space definition (in fact, this is precisely how one *defines* nuclear operator spaces; c.f [**EOR**]). 1.4 is equivalent to the original formulation: *$\mathcal{A}$ is nuclear provided there is exactly one (pre) C^*-norm on $\mathcal{A} \otimes \mathcal{B}$ for all C^*-algebras $\mathcal{B}$.* (This equivalence is due to Choi-Effros and Kirchberg (independently) in the completely positive setting; the refinement to completely contractive maps as above is due to Smith. For detailed references, see [**W**].) Nuclear C^*-algebras include the following family.

Definition 1.5. *A C^*-algebra $\mathcal{A}$ is called type I if every irreducible $*$-representative φ of $\mathcal{A}$ on a Hilbert space H satisfies: $K(H) \subset \varphi(\mathcal{A})$.*

(Recall that $\varphi : \mathcal{A} \to B(H)$ is a $*$-representation if φ is an algebraic $*$-homeomorphism. φ is called irreducible if $\varphi(\mathcal{A})$ has no invariant (closed linear) subspaces other than $\{0\}$ and H.)

It should be pointed out that infinite dimensional type I C^*-algebras are never type I von Neumann algebras. Of course von Neumann algebras are also C^*-algebras, but their topology is really the weak*-topology. In fact, an older definition, later proved equivalent: A C^* algebra $\mathcal{A}$ is type I if and only if every factor representation of $\mathcal{A}$ induces a type I von Neumann algebra (which must then simply be $B(H)$ for some H). It is also a theorem that a type I von Neumann algebra is an ℓ^∞-direct sum of algebras of the form $L^\infty(\mu, B(H))$; where μ is a measure with $(L_1(\mu))^* = L^\infty(\mu)$ and $L^\infty(\mu, B(H))$ denotes the bounded μ-measurable $B(H)$-valued (equivalence classes of) functions on the ambient measure space. Now if $\mathcal{A}$ is a C^*-algebra, $\mathcal{A}^{**}$ is a von Neumann algebra; it is then the case that $\mathcal{A}$ *is a type I C^*-algebra if and only if $\mathcal{A}^{**}$ is a type I von Neumann algebra.*

This is, in turn, a special witness to the deep results of Connes [**C**] and Choi-Effros [**CE2**] that a C^*-algebra $\mathcal{A}$ is nuclear if and only if $\mathcal{A}^{**}$ is injective (as defined in Section 2).

Remarks. It is a standard result that every C^*-subalgebra of a type I C^*-algebra is also type I. However C^*-subalgebras of nuclear C^*-algebras need not be nuclear. Profound work of Kirchberg yields that these coincide with the class of exact C^*-algebras, also introduced by him [**Ki2**]. It follows from the definition that nuclear C^*-algebras have the Banach metric approximation property; in fact, their duals *also* have this property. there are known natural examples of exact non-nuclear C^*-algebras with the metric approximation property, but it is unknown if there exist such algebras which *fail* the metric or even the general (unbounded) approximation property for Banach spaces. The following question is also open: *Suppose $\mathcal{A}$ is a C^*-algebra whose dual has the approximation property. Is $\mathcal{A}$ nuclear?*

The CAR, or Fermion algebra is a fundamental example of a non type I nuclear C^*-algebra. This may be defined as the "infinite" tensor product $\overset{\infty}{\underset{1}{\otimes}} M_2$ of the 2×2 matrices. We prefer the following intuitive description.

We identify $B(\ell_2)$ with infinite matrices and define CAR_d to be all $T \in B(\ell_2)$ so that there exist an $n \geq 0$ and an $A \in M_{2^n}$ with

$$T = \begin{bmatrix} A & & & \\ & A & & \\ & & A & \\ & & & \ddots \end{bmatrix} \tag{1.4}$$

It follows that T is indeed bounded; in fact $\|T\| = \|A\|$. We also define $\mathcal{A}_n$ to be all T as in (1.4) with $A \in M_{2^n}$. Then it follows that CAR_d is a unital $*$-subalgebra $*$-isomorphic to M_{2^n}, with

$$\mathrm{CAR}_d = \cup_{n=0}^{\infty} \mathcal{A}_n \ . \tag{1.5}$$

We now define the CAR algebra $\mathbf{A}$ to be the norm-closure of CAR_d. It then follows that $\mathbf{A}$ acts irreducibly on ℓ_2, and moreover, $\mathbf{A}$ has no non-zero compact operators, so $\mathbf{A}$ is clearly not type I. To see that $\mathbf{A}$ is nuclear, for each n, let $i_n : M_{2^n} \to \mathcal{A}_n$ be the $*$-isomorphism given by (1.4), and define a map $\pi_n : \mathcal{A}_{n+1} \to \mathcal{A}_n$ by

$$\begin{cases} \pi_n T = i_n \begin{pmatrix} \frac{A+D}{2} & 0 \\ 0 & \frac{A+D}{2} \end{pmatrix} & \text{if} \\ T = i_{n+1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{cases} \tag{1.6}$$

where A, B, C, D are in M_{2^n}.

It follows easily that π_n is a completely contractive projection mapping $\mathcal{A}_{n+1}$ onto $\mathcal{A}_n$. But then by (1.5), there exists a unique completely contractive surjective projection $P_n : \mathbf{A} \to \mathcal{A}_n$ so that

$$P_n(T) = \pi_n \cdots \pi_{m-1}\pi_m T \tag{1.7}$$

whenever $T \in \mathcal{A}_m$ with $m > n$. Of course then $P_n x \to x$ for all $x \in \mathbf{A}$, showing $\mathbf{A}$ is indeed nuclear.

The CAR algebra, $\mathbf{A}$ is quite different from type I C^*-algebras. In fact it follows from the results of [**HRS**] that if $\mathcal{B}$ is a type I C^*-algebra, then $\mathbf{A}^*$ is not Banach isomorphic (i.e. linearly homeomorphic) to a subspace of $\mathcal{B}^*$, hence $\mathcal{A}$ and $\mathcal{B}$ are not Banach isomorphic. (Here $\mathcal{B}^*$ denotes the Banach space dual of $\mathcal{B}$.) A remarkable result of J. Glimm yields that $\mathcal{A}$ is a universal witness to a C^*-algebra being non-type I.

Theorem 1.6. [**G**]*; see also Theorem 6.73 of* [**Ped**]*) Let $\mathcal{A}$ be a non-type I C^*-algebra. Then the CAR algebra is $*$-isomorphic to a C^*-subquotient of $\mathcal{A}$. That is, there is a C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that* $\mathbf{A}$ *is $*$-isomorphic to a quotient algebra of $\mathcal{B}$.*

Now actually, if one applies the refined formulation given in [**Ped**] and a lifting theorem due to Choi-Effros [**CE1**], one obtains (as observed in [**Ki1**])

Corollary 1.7. *Let $\mathcal{A}$ be a separable non-type I C^*-algebra. Then the CAR algebra is completely isometric to a completely contractively complemented subspace of $\mathcal{A}$.*

Proof. According to Theorem 6.73 of [**Ped**], one may choose q a projection in $\mathcal{A}^{**}$ and $\mathcal{B}$ a C^*-subalgebra of $\mathcal{A}$ so that q commutes with $\mathcal{B}$, $q\mathcal{B}$ is $*$-isomorphic to $\mathbf{A}$ (the CAR algebra), and

$$q\mathcal{B} = q\mathcal{A}q \ . \tag{1.8}$$

Since $q\mathcal{B}$ is thus nuclear, by the lifting result in [**CE1**], there exists a complete contraction $L : q\mathcal{B} \to \mathcal{B}$ so that

$$\pi L = I_{q\mathcal{B}} \tag{1.9}$$

where $\pi \to q\mathcal{B}$ is the quotient map given by $\pi b = qb$ for all $b \in \mathcal{B}$. Then define $U : \mathcal{A} \to q\mathcal{B}$ by

$$Ua = qaq \quad \text{for all } a \in \mathcal{A}. \tag{1.10}$$

Finally, *define* W by

$$W = L(q\mathcal{B}) \ . \tag{1.11}$$

Then for all $w \in W$,

$$\begin{aligned} LU(w) &= L(qwq) \\ &= L(qw) \quad \text{because } q \text{ commutes with } \mathcal{B} \\ &= w \quad \text{by (1.9).} \end{aligned} \tag{1.12}$$

It then follows, since of course U is a complete contraction, that W is completely isometric to $\mathbf{A}$ *and* W is completely contractively complemented in $\mathcal{A}$ via the projection $P = LU$. □

It is a famous discovery in Banach space theory, due to A. Milutin [**M**], that for any compact metric space K, $C(K)$ is isometric to a contractively complemented subspace of $C(\mathbf{D})$, where $\mathbf{D}$ denotes the Cantor discontinuum. Milutin then deduces that if K is uncountable, $C(K)$ is isomorphic to $C(\mathbf{D})$. E. Kirchberg has established the quantized version of this result, proving the following converse.

THEOREM 1.8. [**Ki1**] *Let $\mathcal{A}$ be a separable nuclear C^*-algebra. Then $\mathcal{A}$ is completely isometric to a completely contractively complemented subspace of the CAR algebra.*

A simple application of the operator space version of the Pełczyński decomposition method ([**Pe1**]) then yields

COROLLARY 1.9. [**Ki1**] *Any separable non-type I nuclear C^*-algebra is completely isomorphic to the CAR-algebra.*

PROOF. For operator spaces X and Y, let $X \overset{cc}{\hookrightarrow} Y$ denote: X is completely isometric to a completely contractively complemented subspace of Y. Now let $\mathcal{D} = (\mathbf{A} \oplus \mathbf{A} \oplus \cdots)_{c_0}$. Of course $\mathcal{D}$ is a nuclear C^*-algebra. Now if $\mathcal{B}$ is a separable non-type I nuclear C^*-algebra, then by Corollary 1.7

$$\mathcal{B} \overset{cc}{\hookrightarrow} \mathcal{D} \tag{1.13}$$

(since $\mathcal{B} \overset{cc}{\hookrightarrow} \mathbf{A}$ and trivially $\mathbf{A} \overset{cc}{\hookrightarrow} \mathcal{D}$). But by Corollary 1.7 and Theorem 1.8,

$$\mathcal{D} \overset{cc}{\hookrightarrow} \mathbf{A} \overset{cc}{\hookrightarrow} \mathcal{B} \ . \tag{1.14}$$

It then follows by the Pełczyński decomposition method that $\mathcal{B}$ is completely isomorphic to $\mathcal{D}$, whence also $\mathcal{D}$ is completely isomorphic to $\mathbf{A}$. □

This leaves the following open question:

Problem. Classify the separable (infinite-dimensional) type I C^*-algebras to complete isomorphism.

I believe the classification should be the *same* as the Banach classification. However I also believe the answer is far more intricate than the known commutative case; see [**Ro2**] for an exposition of the latter.

2. The CSEP and the CSCP

We are concerned here with quantized versions of the Separable Extension Property for Banach spaces, which is defined as follows.

Definition 2.1. *A Banach space Z has the Separable Extension Property (SEP) provided for all separable Banach spaces $X \subset Y$ and bounded linear operators $T : X \to Z$, there exists a bounded linear operator $\tilde{T} : Y \to Z$ extending T. That is, we have the diagram*

$$\begin{tikzcd} Y \arrow[dr, "\tilde{T}"] & \\ \bigcup & \\ X \arrow[r, "T"] & Z \end{tikzcd} \tag{2.1}$$

If $\lambda \geq 1$ is such that $\tilde{T}$ can always be chosen with $\|\tilde{T}\| \leq \lambda \|T\|$, we say that Z has the λ-SEP.

Our first quantized version of the SEP goes as follows.

Definition 2.2. *An operator space Z has the Complete Separable Extension Property (the CSEP) provided for all separable operator spaces $X \subset Y$ and completely bounded $T : X \to Z$, there exists a completely bounded $\tilde{T} : Y \to Z$ extending T. That is. we have that (2.1) holds for completely bounded maps. Again, if $\tilde{T}$ can always be chosen with $\|\tilde{T}\|_{cb} \leq \lambda \|T\|_{cb}$, we say that Z has the λ-CSEP.*

It turns out that if Z has the CSEP, then Z has the λ-CSEP for some λ, and of course a similar statement holds for the SEP itself. We are mainly interested in the case of separable Z. Sobczyk proved in 1941 that c_0 has the 2-SEP and 2 is best possible here [**Sob**]; in fact, if Z is infinite-dimensional separable with the λ-SEP, then $\lambda \geq 2$. Zippin proved in 1977 the far deeper converse; *every infinite dimensional separable Banach space with the SEP is isomorphic to* c_0 [**Z**].

We first give a proof of Sobczyk's theorem which motivates the approach to the quantized versions of the SEP given in [**AR**]. We recall that a Banach space Z is called *isomorphically injective* if for all Banach spaces $X \subset Y$ and operators $T : X \to Z$, there exists a $\tilde{T}$ satisfying (2.1). If we require also that we can choose $\tilde{T}$ with $\|\tilde{T}\| = \|T\|$, we say that Z is *isometrically injective*. Similarly, an operator space Z is called *isomorphically injective* if for all operator spaces $X \subset Y$ and completely bounded $T : X \to Y$, there is a completely bounded $\tilde{T}$ satisfying (2.1); if again we require that $\tilde{T}$ can be chosen with $\|\tilde{T}\|_{\mathrm{cb}} = \|T\|_{\mathrm{cb}}$, Z is called *isometrically injective.* (In the literature, isometrically injective operator spaces are just termed injective.) It is easily seen, from the Hahn-Banach theorem, that ℓ^∞ is isometrically injective. The non-commutative version of this is true (throughout, we use complex scalars).

THEOREM 2.3. *$B(H)$ is an isometrically injective operator space, for any Hilbert space H.*

This was proved for the case of completely positive maps and self-adjoint operator spaces in the domain by Arveson [**A**], and later in general by Paulsen (cf. [**Pa**]) and Wittstock [**Wi**]. It follows from 2.3 that *a separable operator space has the CSEP if and only if it is completely complemented in every separable operator superspace.* Similarly, an operator space is isomorphically injective if and only if it is completely complemented in every operator superspace.

The following result is an immediate consequence of Corollaries 1.2 and 1.3.

COROLLARY 2.4. *Let X be a non-reflexive operator space. If X is separable with the CSEP, X contains a subspace completely isomorphic to c_0. If X is isomorphically injective, X contains a subspace completely isomorphic to ℓ^∞.*

It is known that a Banach space is isometrically injective if and only if it is isometric to $C(\Omega)$ for some extremely disconnected compact Hausdorff space Ω. It is a famous open problem if every isomorphically injective Banach space is isomorphic to an isometrically injective one. Similarly, we have the quantized version: *Is every isomorphically injective operator space completely isomorphic to an isometrically injective operator space?*

We next deal with the proof of Sobczyk's theorem, which motivates the approach to the quantized versions given in [**AR**]. We first formulate complementation in terms of lifts.

DEFINITION 2.5. *Let $X \subset Y$ be Banach (resp. operator) spaces and let $\pi : Y \to Y/X$ be the quotient map. A bounded linear map $L : Y/X \to Y$ is a lift of $I_{Y/X}$ if $I_{Y/X} = \pi L$. That is, the following diagram holds:*

$$
\begin{tikzcd}
 & Y \arrow[d, "\pi"] \\
Y/X \arrow[ur, "L"] \arrow[r, "I"] & Y/X
\end{tikzcd}
\tag{2.2}
$$

Then it is easily seen that X is complemented in Y if and only if $I_{Y/X}$ admits a lift. In fact, we have the following simple result.

Proposition 2.6. *Let $X \subset Y$ be Banach (resp. operator) spaces, and let $\lambda \geq 1$. Then X is λ-co-complemented in Y (resp. λ-completely co-complemented in Y) if and only if $I_{Y/X}$ admits a lift L with*

$$\|L\| \leq \lambda \quad (resp.\ \|L\|_{cb} \leq \lambda). \tag{2.3}$$

Proof. Suppose L satisfies (2.2). Then setting $Q = L\pi$, we easily verify that Q is a projection on Y with kernel equal to X, and of course $\|Q\| \leq \|L\|$ (resp. $\|Q\|_{\mathrm{cb}} \leq \|L\|_{\mathrm{cb}}$). Conversely, if $Q : Y \to Y$ is a bounded linear projection with $X = \operatorname{Ker} Q$, let $W = Q(Y)$. It follows easily that $\pi|W$ maps W onto Y/X and

$$\|\pi w\| \geq \frac{1}{\|Q\|}\|w\| \qquad \text{for all } w \in W. \tag{2.4}$$

Then $L = (\pi|W)^{-1}$ is the desired lift; moreover in the completely bounded setting, $\|L\|_{\mathrm{cb}} \leq \|Q\|_{\mathrm{cb}}$. $\square$

Next we recall a classical theorem of Borsuk [**B**], which asserts that if K is a compact metrizable subset of a compact Hausdorff space Ω, then there is a *linear* operator $L : C(K) \to C(\Omega)$ of norm one so that $(Lf)|K = f$ for all $f \in K$. This may be reformulated in the language of C^*-algebras as follows.

Theorem 2.7. *Let $\mathcal{A}$ be a unital commutative C^*-algebra and $\mathcal{J}$ be a (closed) ideal in $\mathcal{A}$ so that $\mathcal{A}|\mathcal{J}$ is separable. Then there exists a contractive lift $\mathcal{A}|\mathcal{J} \to \mathcal{A}$ of $I_{\mathcal{A}|\mathcal{J}}$.*

Proof. By the Gelfand-Naimark theorem, $\mathcal{A}$ is $*$-isometric to $C(\Omega)$ for some compact Hausdorff space Ω, and moreover, if we just assume $\mathcal{A} = C(\Omega)$, then for some closed subset K of Ω, $\mathcal{J} = \{f \in \mathcal{A} : f(k) = 0 \text{ for all } k \in K\}$. $\mathcal{A}|\mathcal{J}$ is separable if and only if K is metrizable; thus 2.6 is just a reformulation of Borsuk's theorem. $\square$

Now we give a

Proof of Sobczyk's Theorem. We first note that c_0 is an *ideal* in ℓ^∞, which can be regarded as a C^*-algebra. Let then $X \subset Y$ be separable Banach spaces and $T : X \to c_0$ be a given operator. Since ℓ^∞ is isometrically injective, we may choose $\tilde{T} : Y \to \ell^\infty$ extending T with $\|\tilde{T}\| = \|T\|$. Let $\mathcal{A}$ be the (commutative) C^*-subalgebra of ℓ^∞ generated by c_0 and $T(Y)$. Then of course c_0 is an ideal in $\mathcal{A}$ also, and hence is co-contractively complemented in $\mathcal{A}$ by Theorem 2.7 and Proposition 2.6; that is, we may choose a projection $P : \mathcal{A} \to c_0$ with $\|I - P\| = 1$; hence $\|P\| \leq 2$. But then letting $S = P\tilde{T}$, S is an extension of T to Y, and $\|S\| \leq 2$. Thus c_0 has the 2-SEP. $\square$

The remainder of this section deals with *quantized* versions of this argument. Before going into this, we note that it is an open question if the conclusion of Theorem 2.7 holds if we drop the assumption that $\mathcal{A}$ is commutative. In fact, it is open, if every ideal of a separable C^*-algebra $\mathcal{A}$

is *complemented* in $\mathcal{A}$; i.e. if there exists a bounded linear lift L of $I_{\mathcal{A}/\mathcal{J}}$. Deep work of Ando [**A**] yields that the answer is affirmative, however if $\mathcal{A}/\mathcal{J}$ has the bounded approximation property; in particular, if $\mathcal{A}/\mathcal{J}$ has the metric approximation property then $I_{\mathcal{A}/\mathcal{J}}$ admits a contractive lift. An important special case of this problem: *let Y be a separable subspace of $B(\ell_2)$ containing* $\mathbf{K}$*. Is* $\mathbf{K}$ *complemented in* Y*?* Trivially we may assume Y is a C^*-algebra, by just replacing Y by its generated C^*-algebra. An approach to a possible positive resolution to this is given in [**AR**]. The author now guesses, however, this last problem has a negative answer. (A further complement to this problem is given by Proposition 2.20 below.)

We now summarize the main known results on separable spaces with the CSEP. Let R (resp. C) denote the ROW (resp. COLUMN) operator space. Identifying $B(\ell_2)$ with infinite matrices, R is simply all such matrices with non-zero entries only in the first row. Similarly, C is all matrices with non-zero entries only in the first column. It is easily seen that R and C are isometric to ℓ_2; their matrix representation inside $B(\ell_2)$ determines their operator space structure. More generally, if $1 \leq j, k, \leq \infty$, $M_{j,k}$ denotes all operators in $B(\ell_2)$ whose matrices have non-zero entries only in their first j rows and k columns. Thus $M_{\infty,\infty} = B(\ell_2)$, $M_{1,\infty} = R$, $M_{\infty,1} = C$.

If X, Y are operator spaces, $X \oplus Y$ denotes their ℓ^∞ direct sum. If $X_1, X_2, \ldots$ are given operator spaces, $(X_1 \oplus X_2 \oplus \cdots)_{c_0}$ denotes their c_0-direct sum; i.e. the Banach space of all sequences (x_n) with $x_n \in X_n$ for all n and $\|x_n\| \to 0$; $(X_1 \oplus X_2 \oplus \cdots)_{\ell^\infty}$ denotes their ℓ^∞-direct sum. Both of these spaces are just endowed with the corresponding ℓ^∞-direct sum operator structures. Finally, we denote $(X \oplus X \oplus \cdots)_{c_0}$ by $c_0(X)$ and $(X \oplus X \oplus \cdots)_{\ell^\infty}$ by $\ell^\infty(X)$. The following is the main "positive" result on the CSEP.

Theorem 2.8. *For all $n \geq 1$, $c_0(M_{n,\infty} \oplus M_{\infty,n})$ has the 2-CSEP.*

(This is established in [**Ro2**] with "$2+\varepsilon$"in place of "2", $\varepsilon > 0$ arbitrary. The refinement eliminating $\varepsilon > 0$ is given in [**AR**].)

Now for each $n < \infty$, $c_0(M_{n,\infty})$ is completely isomorphic to $c_0(R)$ and $c_0(M_{\infty,n})$ is completely isomorphic to $c_0(C)$; however, just the Banach-Mazur distance itself of $c_0(R)$ to $c_0(M_{n,\infty})$ leads to infinity; i.e. $d(c_0(\ell_2), c_0(M_{n,\infty})) \to \infty$ as $n \to \infty$. Thus it appears surprising that the λ-CSEP constant in 2.8 is best possible, namely $\lambda = 2$.

Conjecture. *Let X be a separable operator space with the CSEP. Then X is completely isomorphic to a subspace of $c_0(R \oplus C)$.*

Of course any completely complemented subspace of $c_0(R \oplus C)$ has the CSEP; naturally this includes c_0 itself. See Conjecture 4.6 of [**Ro2**] for a list of 21 completely complemented operator subspaces of $c_0(R \oplus C)$, which could conceivably be the entire family of separable infinite-dimensional ones with the CSEP.

Now $R \oplus C$ itself is isometrically injective (as an operator space), so has the CSEP. As shown in Proposition 22 of [**Ro2**], any reflexive operator space with the CSEP is isomorphically injective. A problem thus related to the above conjecture: *is every separable (infinite-dimensional) isomorphically injective operator space completely isomorphic to R, C or $R \oplus C$?* The problem has been answered affirmatively by Robertson in case the operator space is actually isometrically injective [**R**].

Of course, $M_{n,\infty} \oplus M_{\infty,n}$ is completely isometric to a subspace of $\mathbf{K}$ for all n. However $\mathbf{K}$ itself *fails* the CSEP. This result is due to Kirchberg [**Ki2**]. In fact, he obtains that $\mathbf{K}_0$ fails the CSEP, where

$$\mathbf{K}_0 = (M_1 \oplus M_2 \oplus \cdots \oplus M_n \oplus \cdots)_{c_0} \ . \tag{2.5}$$

Here, we regard $\mathbf{K}_0$ as a C^*-algebra; notice that $\mathbf{K}_0$ is an ideal in the "finite" von Neumann algebra $\mathcal{N} = (M_1 \oplus M_2 \oplus \cdots)_\infty$.

A new proof that $\mathbf{K}_0$ fails the CSEP is given in [**OR**], via the following result (See Corollary 4.9 of [**OR**]).

Theorem 2.9. *There exists an operator space Y containing $\mathbf{K}_0$ so that $Y/\mathbf{K}_0$ is completely isometric to c_0 and $\mathbf{K}_0$ is completely uncomplemented in Y.*

The particular construction in [**OR**] yields that $\mathbf{K}_0$ is Banach cocontractively complemented in Y The same holds for Kirchberg's construction mentioned above; hence these results cannot resolve the complemented ideal problem mentioned earlier. (Actually we show later that this is a consequence of a general principle; see Proposition 2.20.)

Is there a quantized version of the SEP which $\mathbf{K}$ satisfies? For $\mathbf{K}$ is often thought of as quantized c_0. There is indeed; the "culprit" in the counter example Y of 2.9: Y fails to be *locally reflexive* as an operator space.

Definition 2.10. *An operator space X is called* locally reflexive *if there is a $\lambda \ge 1$ that for all $\varepsilon > 0$ and finite dimensional subspaces F and G of X^* and X^{**} respectively, there exists a linear operator $T : G \to X$ satisfying*

$$\langle Tg, f\rangle = \langle g, f\rangle \quad \text{for all } g \text{ in } G \text{ and } f \in F \tag{2.6}$$

and

$$\|T\|_{cb} < \lambda + \varepsilon \ . \tag{2.7}$$

If λ works, X is called λ-locally reflexive.

If X is any Banach space, then X is 1-locally reflexive and hence (X, MIN) is 1-locally reflexive. Remarkable permanence properties yield that if X is λ-locally reflexive, so is any subspace (cf. [**ER**], [**Pi**]). A C^*-algebra is either 1-locally reflexive or non-locally reflexive [**EH**]. Nuclear C^*-algebras are locally reflexive, but for example $B(H)$ is *not* (for H infinite-dimensional).

As noted above, a separable operator space has the CSEP provided it is completely complemented in every separable operator superspace.

Definition 2.11. *A separable locally reflexive operator space Z has the Complete Separable Complementation Property (the CSCP) provided every* *complete isomorph of Z is completely complemented in every separable locally reflexive operator superspace. Equivalently, given $X \subset Y$ separable locally reflexive operator spaces and $T : X \to Z$ a complete surjective homomorphism, there exists a completely bounded $\tilde{T}$ satisfying (2.1).*

This concept was introduced in [**Ro2**], where it was established that $\mathbf{K}_0$ has the CSCP. Subsequently, T. Oikhberg and the author jointly established the following result.

THEOREM 2.12. [**OR**] $\mathbf{K}$ *has the CSCP.*

Now it follows that then every completely complemented subspace of $\mathbf{K}$ has the CSCP. An affirmative answer to the following would extend Zippman's result [**Z**].

CONJECTURE. *Every operator space with the CSCP is completely isomorphic to a subspace of* $\mathbf{K}$.

For a possible list of all *primary* completely complemented subspaces of $\mathbf{K}$, see Conjecture 4.10 of [**Ro2**]. There are 11 such candidates. Taking direct sums of these yields a finite (up to complete isomorphism) family of operator spaces which could conceivably *be* the list of all operator spaces with the CSCP. Such a result, if true, would involve many deep new ideas.

The following equivalences for the CSCP are established in [**OR**].

THEOREM 2.13. *Let X be a separable locally reflexive operator space. Then the following are equivalent*

(a) *X has the CSCP.*
(b) *X is completely complemented in every separable locally reflexive superspace.*
(c) *Assuming $X \subset B(H)$, then X is completely complemented in Y for all separable locally reflexive Y with $X \subset Y \subset B(H)$.*

Let us note that it is *false* that spaces Z with the CSCP enjoy the extension property given in (2.1) for completely bounded maps, even if Y is separable locally reflexive. Indeed it is established in [**OR**] that *if $Y = C_1$ (the trace class operators) and for all $X \subset Y$ and completely bounded $T : X \to Z$, there is a completely bounded $\tilde{T} : Y \to Z$ satisfying (2.1), then Z has the CSEP.* However, spaces with the CSCP do satisfy a form of this principle (Theorem 1.6(d) of [**OR**]).

PROPOSITION 2.14. *Let Z have the CSCP and $X \subset Y$ be separable locally reflexive such that X is locally complemented in Y. Then for every completely bounded $T : X \to Z$, there is a completely bounded $\tilde{T} : Y \to Z$ extending T (i.e., so that (2.1) holds).*

Recall that if X and Y are operator spaces, with $X \subset Y$, then X is called *locally complemented in Y* if there is a $C < \infty$ so that X is C-completely complemented in Z for all $X \subset Z \subset Y$ with Z/X finite dimensional. It is proved in [**Ro2**] that if Y is locally reflexive, then X is locally complemented in Y if and only if X^{**} is completely complemented in Y^{**}.

As noted above, there exist completely bounded operators from subspaces of C_1 into $\mathbf{K}$ with no completely bounded extensions. However the following problem *may* have an affirmative answer.

PROBLEM. Let $\mathcal{A}$ be a separable nuclear C^*-algebra, X a subspace, and $T : X \to \mathbf{K}$ be a completely bounded map. Does there exist a completely bounded extension $\tilde{T} : \mathcal{A} \to \mathbf{K}$?

We finally sketch a route throough Theorems 2.8 and 2.12, following the approach in [**AR**]. The following concept is fundamental.

DEFINITION 2.15. *Let $X \subset Y$ be Banach/operator spaces.*

(a) *X is called an M-summand in Y if there exists a closed linear subspace Z of Y with $X \oplus Z = Y$ so that*

$$\|x + z\| = \max\{\|x\|, \|z\|\} \quad \textit{for all } x \in X \textit{ and } z \in Z. \tag{2.8}$$

In the operator space case, X is called a complete M-summand if Z can also be chosen so that

$$\|(x_{ij} + z_{ij})\| = \max\{\|(x_{ij})\|, \|(z_{ij})\|\} \tag{2.9}$$

for all n and $n \times n$ matrices (x_{ij}) and (z_{ij}), of elements of X and Z respectively.

(b) *X is called an M-ideal (resp. complete M-ideal) in Y if $X^{**} = X^{\perp\perp}$ is an M-summand (resp. complete M-summand) in Y^{**}.*

It is a remarkable theorem that *if $\mathcal{A}$ is a C^*-algebra and $X \subset \mathcal{A}$, then X is an M-ideal iff X is an algebraic (closed 2-sided) ideal iff X is a complete M-ideal.* This is due to Alfsen-Effros [**AE**] and Smith-Ward [**SW**]. The entire idea of M-ideals appears in the seminal work in [**AE**]. The following isometric lifting result is established in the appendix to [**AR**], extending a lifting result in [**EH**] to the pure operator spaces setting. (An operator space X is called *nuclear* if it satisfies Definition 1.4, replacing "$\mathcal{A}$" by "X" in 1.4.)

THEOREM 2.16. *Let $X \subset Y$ be operator spaces with X a nuclear complete M-ideal in Y, Y locally reflexive, and Y/X separable. Then $I_{Y/X}$ admits a completely contractively life $L : Y/X \to Y$.*

We need one more concept, which appears fundamental for the study of the CSEP.

DEFINITION 2.17. *An operator space X is said to be of finite matrix type if there is a $C \geq 1$ so* that for any finite dimensional operator space G, there is an integer n with

$$\|T\|_{cb} \leq C\|T\|_n \quad \textit{for all linear operators } T : G \to X. \tag{2.10}$$

If C works, we say X is C-finite.

This concept was introduced in [**Ro2**]. In [**OR**], it is proved that *if X is separable and $c_0(X)$ has the CSEP, then X is of finite matrix type.* This suggests the

CONJECTURE. *If X is separable with the CSEP, then X is of finite matrix type.*

It also shows that $\mathbf{K}_0$ fails the CSEP, since it is easily seen that $\mathbf{K}_0$ in *not* of finite matrix type. (In fact, Theorem 2.9 is proved in the course establishing this result). The following yields the needed remaining ingredients for Theorem 2.8.

PROPOSITION 2.18.

(a) *If an operator space X is C-finite, so is $\ell^\infty(X)$.*
(b) *If X is C-finite, X is C-locally reflexive.*
(c) *$M_{n,\infty} \oplus M_{\infty,n}$ is 1-finite, for all n.*
(d) *$c_0(M_{n,\infty} \oplus M_{\infty,n})$ is a complete M-ideal in $\ell^\infty(M_{n,\infty} \oplus M_{\infty,n})$, for all n.*

(a),(b) are proved in [**AR**] and (c) is established in [**Ro2**]. Also, (d) is proved in [**AR**], as a simple consequence of the criterion developed there for X to be a complete M-ideal in Y, namely that existence of a *complete M-approximate identity.*

PROOF OF THEOREM 2.8. Let $n \geq 1$, let $X \subset Y$ be separable operator spaces, let $Z = c_0(M_{n,\infty} \oplus M_{\infty,n})$, and let $T : X \to Z$ be a completely bounded map. It follows easily that setting $W = \ell^\infty(M_{n,\infty} \oplus M_{\infty,n})$, then W is completely contractively complemented in $B(\ell_2)$ and hence W is isometrically injective, by Theorem 2.3. Then there exists an extension $\tilde{T} : Y \to W$ of T with

$$\|\tilde{T}\|_{\mathrm{cb}} = \|T\|_{\mathrm{cb}} \ . \tag{2.11}$$

Now by Proposition 2.18, W is 1-finite and hence 1-locally reflexive. It is easily seen directly that Z is nuclear; of course this also follows since Z is completely isometric to a completely contractively complemented subspace of $\mathbf{K}$. Furthermore, since Z is a complete M-ideal in W, Z is a complete M-ideal in $\tilde{W} \overset{\mathrm{def}}{=} \overline{Z + \tilde{T}(Y)}$. Again, $\tilde{W}$ is 1-locally reflexive, and so by Theorem 2.16, Z is completely co-contractively complemented in $\tilde{W}$. But then there exists a projection P from $\tilde{W}$ onto Z with

$$\|P\|_{\mathrm{cb}} \leq 2 \ . \tag{2.12}$$

Finally, letting $S = P\tilde{T}$, then S is a cb extension of T with $\|S\|_{\mathrm{cb}} \leq 2\|T\|_{\mathrm{cb}}$. $\square$

The following is one of the crucial ingredients needed to establish Theorem 2.12.

THEOREM 2.19. [**OR**] *Let $X \subset Y$ be separable operator spaces, $\tilde{X} \subset B(H)$, and $T : X \to \tilde{X}$ a complete isomorphism from X onto $\tilde{X}$. Then there exists a $\tilde{Y}$ with $\tilde{X} \subset \tilde{Y} \subset B(H)$ and a complete surjective isomorphism $\tilde{T} : Y \to \tilde{Y}$ extending T.*

This result bears on the "complemented ideal" problem discussed following the proof of Sobczyk's theorem above.

PROPOSITION 2.20. *Let $X \subset Y$ be separable operator spaces with X completely isomorphic to $\mathbf{K}$ or $\mathbf{K}_0$. If Y/X has the bounded approximation property, then X is complemented in Y.*

PROOF. Suppose X is completely isomorphic to $\mathbf{K}$. Let $T : X \to \mathbf{K}$ be a complete surjective isomorphism, and let $\mathbf{K} \subset \tilde{Y} \subset B(\ell_2)$ and $\tilde{T} : Y \to \tilde{Y}$ be a complete isomorphism extending T. Then $\tilde{Y}/\mathbf{K}$ is isomorphic to Y/X and so has the bounded approximation property. $\mathbf{K}$ is moreover an M-ideal in $\tilde{Y}$, since it is an ideal in $B(\ell_2)$. By a result of Ando mentioned above [**A**], $\mathbf{K}$ is complemented in $\tilde{Y}$ and hence X is complemented in Y. Now if X is completely isomorphic to $\mathbf{K}_0$, set $W = (M_1 \oplus M_2 \oplus \cdots)_\infty$. It is known that W is completely isomorphic to $B(\ell_2)$. (This is a simple application of the decomposition method for operator spaces). It follows, since Theorem 2.19 is invariant under complete isomorphisms, that one may replace $B(\ell_2)$ by W in its statement. Thus we now let $T : X \to \mathbf{K}_0$ be a complete isomorphism and choose $\mathbf{K}_0 \subset \tilde{Y} \subset W$ and $\tilde{T} : Y \to \tilde{Y}$ a complete isomorphism extending T. Again, $\mathbf{K}_0$ is an M-ideal in $\tilde{Y}$ and $\tilde{Y}/\mathbf{K}_0$ is isomorphic to Y/X and so has the bounded approximation property, so again by [**A**], $\mathbf{K}_0$ is complemented in $\tilde{Y}$ and hence X is complemented in Y. $\square$

REMARKS. A different proof is given in [**AR**] of the special case of Ando's result used above, via the concept of extendable *local liftings* (ell's) introduced there. Thus, given Banach spaces $X \subset Y$, (X, Y) is said to admit ell's if there is a $C \geq 1$ so that for all finite-dimensional $E \subset Y/X$, there exists a $T : Y/X \to Y^{**}$ with $\|T\| \leq C$ so that $T(E) \subset Y$ and $e = \pi T(e)$ where $\pi : Y \to Y/X$ is the quotient map. It is proved in [**AR**] that if $\mathcal{J}$ *is a nuclear ideal in a C^*-algebra $\mathcal{A}$ so that $\mathcal{A}/\mathcal{J}$ is separable, then $\mathcal{J}$ is complemeted in $\mathcal{A}$ provided $(\mathcal{J}, \mathcal{A})$ has extendable local liftings.* (This is easily seen to be the case if $\mathcal{A}/\mathcal{J}$ has the bounded approximation property.) It follows that *the conclusion of 2.20 holds if one replaces the assumption that Y/X has the bounded approximation property by the more general one that (X, Y) has ell's.*

The following result is our last needed ingredient for 2.12.

THEOREM 2.21. [**AR**] *Let $\mathcal{J} \subset Y \subset \mathcal{A}$ with $\mathcal{J}$ a nuclear ideal in a C^*-algebra $\mathcal{A}$ and Y a λ-locally reflexive operator space with $Y/\mathcal{J}$ separable. Then for every $\varepsilon > 0$, there exists a completely bounded lift $L : Y/\mathcal{J} \to Y$ of $I_{Y/\mathcal{J}}$ with $\|L\|_{cb} < \lambda + \varepsilon$.*

This result is generalized to a pure operator space theorem in [**AR**, Theorem 2,4]. The hypotheses then require more than the assumption that $\mathcal{J}$ is a complete M-ideal, for the proof makes use of the special approximate identities that exist in (non-unital) C^*-algebras. Theorem 2.21 generalizes a result of Effros-Haagerup [**EH**], which establishes 2.21 when $Y = \mathcal{A}$ itself. The fact that Y need not be 1-locally reflexive causes difficulties, however, and the proof in [**AR**] is somewhat delicate, although based in part on the techniques in [**EH**]. Note also that the result of [**EH**] itself follows from Theorem 2.16, whose proof in [**AR**] follows Ando's construction rather than the argument in [**EH**]. In fact, we don't know if the conclusion of Theorem 2.16 holds, if "1" and "contractive" are deleted in its statement.

We finally conclude with the

Proof of Theorem 2.12. Let $X \subset Y$ be separable operator spaces with Y locally reflexive and let $T : X \to \mathbf{K}$ be a complete surjective isomorphism. By Theorem 2.19, there exists a $\tilde{Y}$ with $\mathbf{K} \subset \tilde{Y} \subset B(\ell_2)$ and a complete isomorphism $\tilde{T} : Y \to \tilde{Y}$ extending T. But then $\tilde{Y}$ is separable locally reflexive, and of course $\mathbf{K}$ is a nuclear ideal in $\mathcal{A} = B(\ell_2)$. Theorem 2.20 yields a completely bounded projection P from $\tilde{Y}$ onto $\mathbf{K}$, and hence $S = P\tilde{T}$ is the desired extension from Y to $\mathbf{K}$. □

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