Math 302 (Rusin) — Test 4 (Final) Answers — Dec 17 2013

1 I will give you 2 points for each winning Set$^{\texttt{(tm)}}$ hand you can find in the picture displayed in the front of the classroom.

[The cards were as follows.
1. A clear purple squiggly 1
2. A solid green diamond 3
3. A solid red squiggle 3
4. A shaded purple oval 3
5. A clear green oval 2
6. A shaded purple squiggly 3
7. A clear red oval 3
8. A shaded green squiggly 2
9. A shaded purple diamond 1
10. A shaded red squiggly 1
11. A solid purple oval 1
12. A solid green squiggly 2]

Answer: I see 6 winning sets. Here is a methodical way to look for them all.

With so few reds we look first for monochromatic sets; obviously the three reds don't make a set. Among the greens we see no 1s, so a set would have to be all of one number, which then means all 2s. But the three 2s are not a set (two solid, one not). Among the purples we see no 2s so again we would have to have all the same number, and it's not going to be all 3s. But the three 1s (cards 1,9,11) do make a set (same in color and number, different in shape and shading).

Now we look for tricolor sets. With no red 2s and no green 1s we either have to have three different numbers or else use only 3s. The latter would require we use cards 2 and 3 (whose match would then be a purple 3 solid oval) or cards 2 and 7 (matching a purple 3 shaded squiggle – card 6). So cards 2,6,7 give the second winning set.

So we are left looking for trioolor sets with three different numbers. We need only look for (red 1, green 2, purple 3) or (red 3, green 2, purple 1) since the other combinations lack one of the color/number combinations. There are only two (red 1, purple 3) combinations; it's easy to compute the green-2 that is needed for each; $\{6, 8, 10\}$ is the only one that works. Likewise we can check the six (red 3, purple 1) combinations; the winning sets are $\{7, 9, 12\}$, $\{3, 5, 9\}$, and $\{1, 3, 8\}$.

2. The sentence "Every kid in my school watched some movie" is ambiguous; it could mean either of these:

$$A : \forall x \exists y \, (\text{kid } x \text{ watched movie } y) \qquad B : \exists y \forall x \, (\text{kid } x \text{ watched movie } y)$$

Describe the difference in meaning between sentences A and B.

Answer: In A, the movie watched by each kid may be different, e.g. Pat watched *Star wars I*, Chris watched *Star wars I& II*, Sasha watched *Star wars II&III* — the only thing we learn in A is that the sets of movies watched by each kid are never empty.

On the other hand, B says something very special: there is one movie that every student watched. (Maybe they all watched *Star wars I* together!)

Certainly if statement B is true, then A must be true too, but the converse is not correct, as my little example already shows.

3. What is the nth term of the sequence

$$a_1 = 2 \cdot 1, \quad a_2 = 2 \cdot (1+3), \quad a_3 = 2 \cdot (1+3+5), \quad a_4 = 2 \cdot (1+3+5+7), \quad \ldots$$

Answer: The formula is $a_n = 2n^2$.

There are several ways to justify this: you could do a proof by induction, or draw an illustrative picture. You could note that the nth odd number is $2n-1$ so that

$$a_n = 2(1+3+5+\ldots+(2n-1)) = (1+(2n-1)) + (3+(2n-3)) + \ldots + ((2n-1)+1)$$

is the sum of n pairs, each summing to $2n$. Or you could write every odd number as the sum of two consecutive numbers, then recognize two sums-of-consecutive-integers:

$$a_n = 2((0+1)+(1+2)+(2+3)+\ldots+((n-1)+n)) = 2(((n-1)+n/2)+n(n+1)/2) = 2n^2$$

You might instead observe that $\frac{a_n}{2}$ is the sum of n consecutive odd numbers; moving one from each off to the side shows $b_n = \frac{a_n]}{2} - n$ is the sum of n consecutive even numbers starting with zero. But then $\frac{b_n}{2}$ is the sum of $n-1$ consecutive integers, starting with 1, and you (should) know how to add up such a sequence: it'd be $(n-1)((n-1)+1)/2 = n(n-1)/2$. So then b_n itself is $n(n-1)$, i.e. $\frac{a_n]}{2} - n = n(n-1)$. Add n to both sides and double this to get $a_n = 2n^2$.

There are lots of variations you could add to these ideas, too. But simply checking two or three or ten or a thousand values of n is not enough to prove that the result holds for *all* values of n. Ask me some time to give you examples of mathematical phenomena that work for dozens of examples or more, and then fail for the next n!

4. Recall that when we are performing modular arithmetic, we say two integers x and y are *inverses* of each other (modulo N) if $x \cdot y \equiv 1 \pmod{N}$. What is the inverse of 5^{34} modulo 37? (This computation will be much easier if you take advantage of Fermat's Little Theorem.)

Answer: Fermat tells us that $5^37 \equiv 5 \pmod{37}$; divide by 5 (which really means multiplying by the inverse of 5, whatever that inverse happens to be) and we get $5^{36} \equiv 1$. But that left-hand side is $5^{34} \cdot 5^2$, so the inverse of 5^{34} is $5^2 = 25$.

It's true that $5^{34} \equiv 3 \pmod{37}$, though this takes a little time to work out (not too much). But even if you simplify that much, it still takes a few minutes to compute the inverse of 3 (mod 37), and even that relies on a little bit of cleverness! So don't go there — use FLT as above.

5. This year, the NSA has decreed that all its messages will be converted to a 59-symbol alphabet (a blank space is symbol #0, then come the 26 lower-case letters, then

the 26 upper-case letters, then 6 other common symbols, up through symbol #58, which will be the symbol "@"). Also, all encryption this year will be done by multiplying the symbols' numbers by a secret multiplier (mod 59) that changes daily. Agent Boris wishes to send a secret message back to Washington, so he encrypts it using today's multiplier, which happens to be 5, before giving (only) the encrypted message to Agent Natasha, and entrusts her to carry the message back to headquarters. She is trustworthy and never attempts to decrypt the message. But the next day she decides to be extra-secure, and takes this *encrypted* message and encrypts its symbols *again* before sending it back to HQ, using the new day's secret multiplier, which happens to be 12. Did this give even more security to her message? Why or why not?

Answer: Big mistake, Natasha! Boris has already replaced each symbol x with $y = 5x$ (computed modulo 59). If Natasha tries to re-encrypt this she will replace each such y with $12y (\text{mod } 59)$. But $12y = 12(5x) = 60x \equiv 1x = x (\text{mod } 59)$, so in other words she has just replaced each of Boris's encrypted characters with the original character it was trying to hide! — she's carrying a completely unencrypted message back to Washington!

6. How would you write (in ordinary base-ten notation) the number X which is expressed in binary as 10101?

How would you express in binary notation the number three-times-X-plus-one?

Four point bonus: what number is expressed in binary as 1010101010101? (Hint: there's a very good reason I asked you to compute $3X + 1$ in problem 6 ...)

Answer: First, $X = 10101$ in binary means $1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 16 + 4 + 1 = 21$. Then $3X + 1 = 64 = 2^6$ is written 100000 in binary.

But in fact we should have computed $3X + 1$ right in binary notation: multiply 10101 by 11 and you'll get 111111; add 1 and there are 6 consecutive carries to get 100000. Then we'd know how to express that other number, call it Y: just as we just computed, $3Y$ is a string of fourteen 1's, so $3Y + 1$ is a 1 followed by fourteen zeros, i.e. 2^{14}. Thus $Y = (2^{14} - 1)/3 = (16384 - 1)/3 = 5461$.

7a. The number $x = 0.027027027\ldots$ is a repeating decimal expansion, so the real number it represents must be rational. Very well then: write x as a ratio of integers: $x = m/n$

7b. Likewise $y = 1/27$ is rational, so its decimal expansion must be eventually periodic. Well, what is it?

Answer: Moving the decimal point three steps to the right we may write $1000x = 27.027027...$ so if we subtract x itself we see $999x = 27$ and so $x = 27/999$ which equals $1/37$ because we can factor $999 = 27 \cdot 37$.

That same factorization lets us write $1/27 = 37/999$; since $1/999 = 0.001001001\ldots$ (that's easy to discover in several ways) we then have $1/27 = .037037037\ldots$

Since $99999 = 3^2 \cdot 41 \cdot 271$ we similarly get expansions like $1/271 = 0.003690036900369\ldots$. $\blacksquare$ See how many pretty patterns we can find?!

8. Here are three sets of real numbers, described in terms of their ordinary base-10 (decimal) expansions. One set is large but finite, one is infinite but countable, and one has

the same cardinality as the set of all real numbers (hence is uncountably infinite). Tell me which is which. For the one that's finite, give me some integer that's surely bigger than the number of elements in that set. For the one that's countable tell me how to make an ordered (infinitely long) list of all its elements. And finally explain why the remaining set cannot possibly be countable. The three sets are:

A. The set of numbers whose decimal expansion uses only 0's and 1's (e.g. 0.010101...)

B. The set of numbers whose decimal expansion never uses a digit twice (except for omitted zeros). For example, 3.14 is in B but 3.1416 is not and neither is .00314

C. The set of numbers whose decimal expansion is symmetrical around the decimal point (e.g. 1523.3251)

Answer: Set A is uncountable. Each number in that set gives rise to a distinct pattern of 0s and 1s. With those kinds of character strings you can already get the binary "decimal" expansion of every real number between 0 and 1. So the cardinality of A is the same as the cardinality of that interval of the real line, which is the same as the cardinality of all the reals. (You could also use the Cantor diagonalization argument to show that no putative list of elements of A can ever be complete.)

Set B is finite. Since you can't repeat a digit (nor the decimal point), every number in B corresponds to some subset of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, "."\}$ and there are only 2^{11} such subsets (count them with a tree that branches 11 times, asking "Is 0 in the set?", "Is 1 in the set?", etc.) Once you know what symbols are involved there are still many ways to rearrange them but not that many – if there are k symbols there are $k!$ rearrangements, and that's less than 11!. So certainly there are no more than $2^{11} \cdot 11!$ elements in B. (In fact that number is too big by a factor of nearly 1000. We can be a little bit more careful with our counting; there are 108,505,112 ways to rearrange up to 11 symbols. Of these the empty string an "." itself are not really numbers, and strings ending in a decimal point, or in ".0", are the same number as the same strings truncated before the decimal point.)

Set C is countable: just make a list of all the integers, in natural order if you prefer, then put a decimal point after each and then add the mirror image of the decimal digits. So $C = \{0.0, 1.1, 2.2, \ldots 10.01, 11.11, \ldots\}$.

9. The United Nations building in New York City is essentially a giant brick, 300 feet tall, 200 feet wide from east to west, and 100 feet deep from north to south. Spiderman is standing on the ground at the southwest corner of the building. Mary Jane is waiting on the roof, near the northeast corner; specifically she is 40 feet south and 10 feet west of that corner. Spidey is anxious to crawl around the outside of the building to be with her. What's the shortest path he can follow to reach her, and how long is it?

Answer: We should be able to lay out the sides of the building on a flat table so that Spidey's path is a straight line. Start with the roof and draw copies of the four walls as rectangles next to it; two of the walls have Peter on them and we can draw the straight lines to compute the distance traveled. For the other two, now lay out ITS three other neighbors and look again. In this way we can identify several straight-line paths that Peter Parket can take to get to MJ. The shortest crosses the south wall, then the east wall, then the roof; from the Pythagorean theorem the length of this route is $\sqrt{310^2 + 260^2} = 404.6$ feet.

10. On the last test, some of you wanted to treat the letters as some kind of surface. Very well, let's do it! Imagine Macy's has created huge balloons for its Thanksgiving Day parade, in the form of letters; for example the letter `A` could be made from two hot-dog shaped balloons — take a long one, bend it in half, and cut out a disk near the center of each half; then take a short balloon and cut off both ends, and glue each end to one of the circles cut into the long balloon; when you reinflate your creation you will get a smooth surface.

If I make such a balloon letter for each of the following letters, what surfaces will I get? Identify each one as being (topologically equivalent to) a sphere, an n-holed torus (tell me what n you get each time), a Klein Bottle, or a projective plane.

`A B C D E F G H I J K L`

Answer: The letters A and D make a 1-holed torus; B makes a 2-holed torus. All the others will have no holes at all so they're not tori at all (and are by definition embeded in 3-dimensional space so they're not the projective plane or the Klein bottle) so they must be of the same topological type as the sphere.

11. Also on the last test, I asked you to cut up a Klein bottle and tell me what you get. This time let me ask the same question except I ask you to make cuts along the lines indicated below. What will you get after cutting this time?

Answer: The three rectangles can first be glued as the left and right edges indicate, to get two rectangles. What was the center strip then glues into a cylinder, and the other rectangle we just made also glues into a cylinder.

12. I have a jar containing 10 red marbles and 10 blue marbles. (They are all identical except for color.) I reach in and pick a marble at random and set it aside. Then I reach in again and again pick out a marble at random from the ones that remain. What is the probability that the two marbles I have selected will be the same color?

Answer: The second draw matches the first one with a 9/20 probability.

13. An ordinary deck of cards has 52 cards in it. Among other things each card is decorated with one of the 13 symbols

$$\mathrm{A}, 2, 3, 4, 5, 6, 7, 8, 9, 10, \mathrm{J}, \mathrm{Q}, \mathrm{K}$$

(For each of these labels there are four cards bearing that label.) Let's play a card game: you will draw exactly one card at random from the deck, and I'll pay you the number of dollars shown on the label, if the label is a number. If you draw an "A" I'll pay \$1, and if you draw a J, Q, or K, I'll pay you \$10. What would be a fair price for me to charge you to play a round of this game? (I'll play as many rounds as you like, if you'll pay me at least this fair price each time!)

Answer: The expected value is

$$\frac{4}{52} \cdot 1 + \frac{4}{52} \cdot 2 + \cdots + \frac{4}{52} \cdot 9 + \frac{12}{52} \cdot 10$$

which is $(1/13)(1+2+\ldots+9+30) = (1/13)(45+30) = 75/13 = 5.76923...$ So a fair price would be just under \$5.77 per round.

Here's what you need to know about Chemistry to see where Problem #3 comes from.

It is (approximately) true that the electrons in an atom arrange themselves into shells — you can think of the shells as spheres wrapped around each other like layers of an onion, with the *nucleus* of the atom deep inside the innermost shell. We number the shells from the inside outwards, i.e. shell #1 is the smallest, then #2, etc. Now, each electron wanders around but is confined to be in just one shell; and as they do so they follow different kinds of paths, called *orbitals*. There's one kind of path called an s-orbital, three different ones called p-orbitals (they're actually mutually perpendicular), five different ones called d-orbitals, and so on. (The names are assigned a little erratically because of tradition, but there is a definite "order" to them that we will need in a moment: s comes "first", then p, then d, then f, etc., and in this order the numbers of orbitals of each type are the consecutive odd numbers: $1, 3, 5, , \ldots$.) Now, shell #1 contains only an s-orbital; shell #2 contains an s orbital and all three p orbitals; shell #3 contains all s, p, and d orbitals; and so on, so that the complete set of all possible orbitals paths is countable, and we can order them by their shell number and their shape, in the following order:

$$1s,\ 2s, 2p,\ 3s, 3p, 3d,\ 4s, 4p, 4d, 4f,\ \ldots$$

The order is significant for the following reason. If you start with a bare nucleus and then start tossing electrons towards it, the electrons will jump into one of the orbitals in one of the shells, but there are two rules they follow: (1) any given orbital within any given shell cannot hold more than 2 electrons (this is the *Pauli Exclusion Principle*), and (2) each added electron will get into the lowest-ranked orbital that's not already full, following the ordering of the orbitals displayed above. (Actually Nature uses a slightly different ordering of the orbitals but we can ignore that in a Math class!) So after 2 electrons, the first shell is full. The next 6 electrons fill shell #2 — two into the $2s$ orbital, and two each into the three (mutually perpendicular) $2p$ orbitals. The next electrons start filling the orbitals of the third shell, and so on. The interesting thing to chemists is that how an atom behaves chemically depends on how close to being full its outermost shell is. They keep a chart of all the elements, called the *Periodic Table*, and as you read it from left to right along one row, you're looking at atoms whose outermost shell is growing from nearly-empty to completely-full; when you start filling in the next shell, you get atoms that occupy the next row of the periodic table. (The number of electrons in one atom of an element is called the *atomic number* of that element.)

The chemical question is this: what is the pattern that determines how long the rows are in the periodic table? In other words, how many electrons does an atom have altogether when its atoms make n completely-full shells?

From what I've just told you about orbitals and so on, that question is really the same as Problem #3 on the test!