Exam II — some possible answers

1. You could use the quotient rule, but it's easier to note that $h(x) = x^{3/2} - x^{1/2} + 2x^{-1/2}$ so that $h'(x) = (3/2)x^{1/2} - (1/2)x^{-1/2} - x^{-3/2}$.

2. The slope of the tangent line is $\frac{dy}{dx}$ (thinking of y as a function of x). Since $x^2 + 2y^2 = 1$ we can differentiate both sides with respect to x to conclude $2x + 4y\frac{dy}{dx} = 0$ and thus $\frac{dy}{dx} = -\frac{2x}{4y}$. So the points in question have $-\frac{2x}{4y} = 2$, that is, $x = -4y$. (In other words, the points we are looking for lie on a certain line through the origin.)

Since the points we want are also on the ellipse, we can find the points as the intersection of the line and the ellipse. Algebraically that means we want points (x, y) where both $x^2 + 2y^2 = 1$ and $x = -4y$. Such points have $16y^2 + 2y^2 = 1$, i.e. $y = \pm\frac{1}{\sqrt{18}}$. Since $x = -4y$ that means the two points are $(x, y) = (\frac{-4}{\sqrt{18}}, \frac{1}{\sqrt{18}})$ and $(x, y) = (\frac{4}{\sqrt{18}}, \frac{-1}{\sqrt{18}})$

It is instructive to draw a picture showing the ellipse, the line $x = -4y$, the two points of intersection, and the two tangent lines; you should be able to see that the slopes of the latter are 2.

3. Using Chain Rule and Quotient Rule we see

$$h'(1) = f'(g(1))g'(1) - \frac{g(1)f'(1) - f(1)g'(1)}{g(1)^2} = 5 \cdot 6 - \frac{2 \cdot 4 - 3 \cdot 6}{2^2} = \frac{55}{2}$$

4. If $y = e^{rx}$ then $y' = re^{rx}$ and $y'' = r^2e^{rx}$, so the left side of the differential equation is $(r^2 - r - 6)e^{rx}$. Because exponentials are never zero, this product is zero iff $0 = r^2 - r - 6 = (r - 3)(r + 2)$, which happens precisely when $r = 3$ or $r = -2$.

5. The cone will have a height h, a bottom radius r, and a volume V which all change over time; we will view them as functions of time elapsed t. (I will measure r and h in centimeters, V in cubic cm, and t in minutes.) The relationship between V and the others is $V = \frac{1}{3}\pi r^2 h$, and we are told that the diameter $2r$ always equals h, so $r = h/2$ and the total volume is $V = \frac{\pi}{12}h^3$ at all times t. Differentiating w.r.t. t then shows $\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$. On the other hand, we are told that $\frac{dV}{dt}$ is a steady 30 cc/min, and that at a certain moment, $h = 10$cm. Thus, at that moment we have $30 = \frac{\pi}{4}(10)^2\frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{6}{5\pi}$ cm/min.

6. Write the two functions in exponential form: $\sinh(x) = (e^x - e^{-x})/2$ and $\cosh(x) = (e^x + e^{-x})/2$, so that $\frac{\sinh(x)}{\cosh(x)} = \frac{(e^x - e^{-x})}{(e^x + e^{-x})} = \frac{(1 - e^{-2x})}{(1 + e^{-2x})}$. As $x \to \infty$, $e^{-2x} \to 0$ and the whole fraction tends to 1.

7. The linearization of any function Z near any point a is the (linear) function $L(x) = Z(a) + Z'(a)(x - a)$. In our case, $Z(x) = \ln(1 + 3x)$, and $a = 0$, so $Z'(x) = \frac{3}{1+3x}$ and we have $Z(a) = 0$, $Z'(a) = 3$, and thus $L(x) = 0 + 3(x - 0) = 3x$. Then note that $\ln(1.3) = Z(0.1)$ which is then approximated by $L(0.1) = 0.3$. (The actual value is $\ln(1.3) = 0.262...$)

8. First note that the domain of T is the set of positive numbers x, so we will not be concerned with what happens for $x \leq 0$ in what follows.

We compute $T'(x) = \frac{1}{x} - \frac{1}{100}$, which is continuous everywhere on $(0, \infty)$ and is zero precisely when $x = 100$. Testing, say, $x = 1$ and $x = 10^6$ we see $T'(x) > 0$ on $(0, 100)$ and $T'(x) < 0$ on $(100, \infty)$. That is, T increases on the first interval, reaches a maximum at $x = 100$, and then decreases.

The maximum point is at $(100, T(100)) = (100, \ln(100) + 49) \approx (100, 53.6)$. In particular, it's in the first quadrant.

We compute $T''(x) = \frac{-1}{x^2}$, which is clearly always negative, so the graph is concave down everywhere: no inflection points at all. Moreover, since the graph is decreasing and concave down on $(100, \infty)$, there can be no horizontal asymptotes. (Indeed, since the function is concave down everywhere, its graph lies below any of its tangent lines. For example, the tangent line at $x = 200$ is $y = T(200) + T'(200)(x - 200) = (\ln(200) + 47) - \frac{x}{200}$, so the graph of T is under this, and certainly slips into the 4th quadrant before $x = 10,000$ or so.)

There is a vertical asymptote at $x = 0$ because $\lim_{x \to 0^+} \ln(x) = -\infty$. Note that this means $T(x) < 0$ for tiny positive x.

Since $x = 0$ is not in the domain, there are no y-intercepts. But we have shown $T(x)$ is negative for tiny x, positive when $x = 100$, and negative again for huge x. Thus there are two x-intercepts. (The actual locations are approximately $x = 5867$ and $x = 10^{-20}$!)

9. We wish to minimize the distance r from the point (0,3) to the points (x, x^2) on the parabola. It's enough to minimize the square $s = r^2$ of that distance, which is $s = (x - 0)^2 + (x^2 - 3)^3 = x^4 - 5x^2 + 9$. That expression certainly grows without bound when x is very large in magnitude, so there will be a minimum value for s. That in turn can only happen at a critical point, which requires $0 = ds/dx = 4x^3 - 10x = 4x(x^2 - (5/2))$. Thus either $x = 0$ or $x = \pm\sqrt{5/2}$. But at $x = 0$ we find $d^2s/dx^2 = 12x^2 - 10$ is negative, so at $x = 0$ the distance function has a local *maximum*. Thus the minimum occurs at one of the other two points, and indeed they are tied for best: in both cases $s = (5/2)^2 - 5(5/2) + 9$. So the closest points on the parabola are $(x, y) = (\pm\sqrt{5/2}, 5/2)$.

10. (a) $\sec(x) = \frac{1}{\cos(x)}$ is a fraction with a non-vanishing numerator and hence is not zero for any x.

(b) *The exam question was mis-worded!* The theorem which asserts that a function with positive and negative values should also take the value zero is the *Intermediate* Value Theorem. I'm very sorry about the error in wording and will take that into account when grading this question.

Anyway, the Intermediate Value Theorem requires the function to be *continuous* on (in this case) the interval $[0, \pi]$; but the secant function has a discontinuity at $x = \pi/2$, so the IVT has nothing to say about this situation — there is no expectation that a discontinuous function will take on the value zero.