

Math 408C (Rusin): Exam III, Nov 22 2011. Some possible ANSWERS.

1. Compute  $\lim_{x \rightarrow 1} \left( \frac{1}{x-1} - \frac{1}{\ln(x)} \right)$ .

ANS: This is a “ $\infty - \infty$ ” type. Write it instead as

$$\lim_{x \rightarrow 1} \frac{\left( 1 - \frac{x-1}{\ln(x)} \right)}{x-1}$$

and then use L'Hopital's Rule to see that the limit equals

$$\lim_{x \rightarrow 1} \left( -\frac{\ln(x) - x + 1}{\ln(x)^2} \right)$$

Now use L'Hopital's Rule again to write this limit as

$$\begin{aligned} \lim_{x \rightarrow 1} \left( -\frac{(1/x) - 1}{2 \ln(x)/x} \right) \\ = \lim_{x \rightarrow 1} \left( -\frac{1-x}{2 \ln(x)} \right) \end{aligned}$$

One more applications of L'Hopital's Rule makes this into

$$= \lim_{x \rightarrow 1} \left( \frac{1}{2/x} \right) = -1/2$$

2. Suppose  $f$  is the function defined by

$$f(x) = \begin{cases} 2, & \text{if } 0 < x < 1 \\ 1, & \text{if } 1 < x < 2 \\ -1, & \text{if } 2 < x < 3 \end{cases}$$

Sketch the graph of  $f$ . Then sketch the graph of a continuous function  $F$  which is an antiderivative of  $f$  on the interval  $(0, 3)$ .

ANS: The function  $F$  may also be found algebraically as

$$F(x) = \begin{cases} 2x, & \text{if } 0 < x \leq 1 \\ x + 1, & \text{if } 1 < x \leq 2 \\ 5 - x, & \text{if } 2 < x \leq 3 \end{cases}$$

3. Compute  $\int_{-2}^2 (3 + \sqrt{4 - x^2}) dx$ . (Hint: you may wish to sketch the graph of this function first.)

ANS: The graph shows a 4x3 rectangle with a semicircle of radius 2 on top, so the integral is  $12 + (1/2)(\pi 2^2) = 12 + 2\pi$ .

4. Estimate the value of  $\int_1^3 \frac{1}{x^3+1} dx$  by computing a Riemann sum for this integral. Your Riemann sum must have at least 4 summands. Your final answer may be in the form of unsimplified fractions, e.g. " $\frac{2}{3} + \frac{15}{16} + \frac{1}{2}$ " would be a suitable form for an answer.

ANS: You could use 4 intervals of length 1/2, and then make the rectangles' height be determined by the heights at the left endpoints: the heights are the values of  $\frac{1}{x^3+1}$  at  $x = 0, 1/2, 1, 3/2$ . Thus the Riemann sum is

$$(1/2)(1) + (1/2)(1/((1/2)^3 + 1)) + (1/2)(1/2) + (1/2)(1/((3/2)^3 + 1))$$

Incidentally, we are almost to the point where you can evaluate that integral exactly; you will be able to give the true value, which is an impressive

$$\ln(2)/3 - \ln(7)/6 - \sqrt{3}\pi/18 + \sqrt{3} \arctan(5/\sqrt{3})/3$$

5. Use the properties of integrals to explain why  $\int_1^\pi \frac{\sin(x^2)}{x} dx \leq \ln(\pi)$ .

ANS: Since  $\sin(u) \leq 1$  for all  $u$ , our integral is at most equal to

$$\int_1^\pi \frac{1}{x} dx = \ln(x)|_{x=1}^{x=\pi} = \ln(\pi)$$

6. If  $G(x) = \int_{2x}^{x^2} \tan(\sqrt{t}) dt$ , then compute  $G'(x)$ .

ANS: If  $H(x)$  is an antiderivative of  $\tan(\sqrt{x})$  then  $G(x) = H(x^2) - H(2x)$ , so by the Chain Rule,  $G'(x) = H'(x^2) \cdot 2x - H'(2x) \cdot 2$ . But  $H'(x) = \tan(\sqrt{x})$ , so our answer is

$$G'(x) = 2x \tan(\sqrt{x^2}) - 2 \tan(\sqrt{2x})$$

(If you wish you could simplify the  $\sqrt{x^2}$  to  $x$  — the definition of  $G$  doesn't make sense if  $x < 0$  because the integrand is only defined for  $t \geq 0$ ; thus we may assume  $x$  is non-negative and so  $\sqrt{x^2} = |x| = x$ .)

7. Evaluate  $\int_0^4 (4-t)\sqrt{t} dt$ .

ANS: This is  $\int_0^4 (4t^{1/2} - t^{3/2}) dt = \left[ \left(\frac{2}{3}\right)4t^{3/2} - \left(\frac{2}{5}\right)4t^{5/2} \right]_0^4 = \left(\frac{2}{3}\right)4^{5/2} - \left(\frac{2}{5}\right)4^{7/2} = \left(\frac{2}{3}\right)32 - \left(\frac{2}{5}\right)128 = 64/3 - 128/5 = 128/15$

8. Evaluate  $\int_{-\pi/2}^{\pi/2} x \sin(x^2) dx$ .

ANS: Using a  $u$ -substitution  $u = x^2$ , so that  $du = 2x dx$ , this integral becomes  $\int_{\pi^2/4}^{\pi^2/4} \frac{1}{2} \sin(u) du$ . You need compute no further, as the integration now takes place over an interval of width zero! Thus the whole integral is zero. (You can also see this by using the substitution  $u = -x$ : that step ends up showing that our integral equals its own negative; zero is the only such real number!)

9. Find an antiderivative of  $\frac{e^t}{e^t + 3}$

ANS: Perform a  $u$ -substitution in the indefinite integral: Let  $u = e^t$  so that  $\frac{e^t}{e^t + 3} dt = \frac{1}{u+3} du$ . The antiderivative here is  $\ln(u + 3) = \ln(e^t + 3)$ .

10. What is the volume of the portion of the unit sphere  $x^2 + y^2 + z^2 \leq 1$  where  $z \geq \frac{1}{2}$ ? (You could call it the “top half of the northern hemisphere”, although it clearly has less than half the volume even though it has half the height!)

Use the method of disks: use the  $z$ -axis as your “chopping line” and keep track of your slices using a parameter  $t =$  distance down from the north pole. So the volume will be  $\int_0^{1/2} A(t) dt$  where  $A(t)$  is the cross-sectional area of the slice taken at depth  $t$  (which is obviously a disk of some radius  $r$ ). If you draw a cut-away view of the sphere (showing the  $x, y$ -plane) you will see that the radius is the distance from the  $z$ -axis towards the unit circle. You should draw a picture in which you can identify line segments of lengths  $r$ ,  $t$ , and  $1 - t$ ; there’s also a right triangle in there whose hypotenuse has length 1 (because it’s the UNIT sphere). This gives you the Pythagorean relation  $r^2 + (1 - t)^2 = 1$ , or  $r^2 = 2t - t^2$ . Then the cross-sectional area is  $A(t) = \pi r^2 = \pi(2t - t^2)$ . When we integrate from  $t = 0$  to  $t = 1/2$  we get  $\pi(t^2 - \frac{t^3}{3})|_{t=0}^{t=1/2} = \pi(\frac{1}{4} - \frac{1}{24}) = \frac{5\pi}{24}$ . (That means this is only  $5/32$  of the total volume of the sphere – rather a small fraction!)