Math 408D (Rusin) Exam 2, Mar 10, 2011. Here are some suggested answers.

**NOTE** For the most part, what you see here is *not* some kind of "analysis" of the test. What I am showing you here is what I would consider to have been good answers from students. In other words, *this is the kind of thing you're supposed to write.* Notice for example that anything that's supposed to be a number or a function is part of an equation, and individual equations are linked into sentences with words that indicate why they're true ("so", "thus", etc.).

1. Compute the first six terms

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

of the Taylor series for $f(x) = \dfrac{x+2x^2}{1-2x^2}$.

Answer:

$$\begin{aligned} f(x) &= (x+2x^2)\cdot\left(\frac{1}{1-2x^2}\right) \\ &= (x+2x^2)\cdot(1+2x^2+4x^4+\dots) \\ &= (x+2x^2)+(2x^3+4x^4)+(4x^5+8x^6)+\dots \\ &= 0+1x+2x^2+2x^3+4x^4+4x^5+\dots \end{aligned}$$

2. Find an antiderivative $F(x)$ of the function $f(x) = \dfrac{e^x-1}{x}$, expressed as a series.

Answer: Since $e^x = \sum_{n\ge 0} \frac{1}{n!}x^n$, $f(x) = \sum_{n=1}^{\infty} \frac{1}{n!}x^{n-1}$ (we start at $n=1$ because the $n=0$ term cancels the "$-1$"), and so we antidifferentiate term by term to get

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{n!}\frac{x^n}{n} = x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \dots$$

3a. Compute the quadratic Taylor polynomial of $f(x) = x\,e^{-x}$ at $x=0$.

Answer: Since $e^x = \sum_{n\ge 0}\frac{x^n}{n!}$, we easily find

$$f(x) = \sum_{n\ge 0} \frac{(-1)^n\, x^{n+1}}{n!}$$

In particular, the quadratic part is simply $T_2(x) = x - x^2$.

3b. Use your polynomial to estimate a value of $x$ for which $f(x) = \dfrac{3}{16}$. (You may use the Quadratic Formula if you wish, although it is not really necessary.)

Answer: $f(x)$ will at least be *near* $3/16$ if $T_2(x)$ is. So we seek solutions to $0 = x - x^2 - 3/16 = -(4x-1)(4x-3)$; thus $x \approx 0.25$ is a reasonable guess.

NB: When $x = 3/4$, $T_2(x)$ is again $3/16$, but since $3/4$ is farther from 0, we don't expect $f(x)$ to be close to $T_2(x)$. In fact, the actual solutions to $f(x) = 3/16$ are at $0.2378\ldots$ and $2.6476\ldots$.

4. Compute the cubic Taylor polynomial for $g(x) = \sqrt{2}\sin(x)$ around the point $x = \pi/4$. Use this polynomial to estimate the numerical value of $g(\frac{\pi}{4} + 1)$. Use Taylor's Inequality (the Taylor Remainder Estimate) to give an upper bound on the error. (That is, you should be able to say, "I am sure that my estimate differs from the correct value by no more than ...")

$$\begin{array}{rll} g(x) & = \sqrt{2}\sin(x) \quad \text{so} & g(\pi/4) = 1 \\ g'(x) & = \sqrt{2}\cos(x) & g'(\pi/4) = 1 \\ g''(x) & = -\sqrt{2}\sin(x) & g''(\pi/4) = -1 \\ g'''(x) & = -\sqrt{2}\cos(x) & g'''(\pi/4) = -1 \\ g^{(iv)}(x) & = \sqrt{2}\sin(x) & \end{array}$$

So $g(x) = 1 + 1\cdot(x - \frac{\pi}{4}) - \frac{1}{2}\cdot(x - \frac{\pi}{4})^2 - \frac{1}{6}\cdot(x - \frac{\pi}{4})^3 + \ldots$

Thus $g(\frac{\pi}{4} + 1) \approx 1 + 1 - 1/2 - 1/6 = 4/3 = 1.333\ldots$.

The error is (for some value of $b$ between $\pi/4$ and $\pi/4 + 1$)

$$\frac{g^{(iv)}(b)}{4!}\cdot 1^4 \leq \frac{\sqrt{2}}{24}\cdot 1 \leq \frac{3/2}{24} = 1/16 = 0.0625.$$

NB: The actual value of $g(\pi/4 + 1)$ is 1.38177329, which is $4/3 + 0.04844$, so our estimate of the error was just a little generous.

5. A curve is given parametrically by the equations

$$x = t + \frac{1}{t}, \qquad y = t - \frac{1}{t}$$

Describe the curve in Cartesian coordinates (i.e. as the solution set of an equation in $x$ and $y$). (*Hint:* add!)

Answer: Since $x + y = 2t$, the curve may be written simply as

$$x = (\frac{x+y}{2}) + (\frac{2}{x+y}).$$

I will give a little bit of extra credit for a nice description of this curve.

Answer: The equation above simplifies to $2x(x + y) = (x + y)^2 + 4$ and then to $x^2 + 2xy - y^2 = 4$. This is a hyperbola (at the origin) since "$b^2 - 4ac > 0$". You can also spot the behaviour of the curve by noting that for $t$ close to zero the parameterization will give points with $y \approx -x$ and for $t$ far from zero, it gives points with $y \approx x$. So the graph is asymptotic to that pair of crossed lines.

6. Give a parameterization of the ellipse $9x^2 + 16y^2 = 25$ which passes along the curve in a clockwise direction, and which starts (at time $t = 0$) at a point having $y = 0$.

Answer: We are looking for functions $x(t)$ and $y(t)$ that make $(\frac{3x}{5})^2 + (\frac{4y}{5})^2 = 1$ and also have $\frac{4y}{5} = 0$ when $t = 0$. We might try $\frac{3x}{5} = \cos(t)$ and $\frac{4y}{5} = \sin(t)$. But note that for small positive $t$ we'd have $y > 0$ and $x > 0$, putting us in the first quadrant, meaning we're going *counter*-clockwise. So instead we can use something like

$$x = -\frac{5}{3}\cos(t), \qquad y = -\frac{5}{4}\sin(t).$$

7. A certain curve is defined parametrically by the equations

$$x = 5 + 3t^2 \qquad y = 2 + t^2 - t^3$$

Find all values of $t$ for which the curve is concave up.

Answer: By Chain Rule, $\dfrac{df}{dx} = \dfrac{df/dt}{dx/dt}$ for any function $f(x)$. In particular $\dfrac{dy}{dx} = \dfrac{2t - 3t^2}{6t} = \dfrac{1}{3} - \dfrac{1}{2}t$. Then

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{3} - \frac{1}{2}t\right) = \frac{\frac{d}{dt}\left(\frac{1}{3} - \frac{1}{2}t\right)}{(dx/dt)} = \frac{-1/2}{6t} = \frac{-1}{12t}$$

Curve is concave up when $\frac{d^2y}{dx^2} > 0$, i.e. when $t < 0$.

8. How long is the curve defined by the equations

$$x = t^4 - 4.5t^2 + 5, \qquad y = 4t^3 + 2$$

as $t$ ranges from 3 to 5?

$$\begin{aligned}
\text{Answer: } L &= \int_{t=3}^{t=5} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int \sqrt{(4t^3 - 9t)^2 + (12t^2)^2}\, dt \\
&= \int \sqrt{16t^6 + 72t^4 + 81t^2}\, dt \\
&= \int_3^5 (4t^3 + 9t)\, dt \\
&= (t^4 + \frac{9}{2}t^2)|_3^5 = (625 - 81) + \frac{9}{2}(25 - 9) = 552
\end{aligned}$$

NB: I rigged up the problem so that that degree-6 polynomial was a perfect square. Had that not been the case, you would not have been able to find an explicit antiderivative: for general polynomials $p(t)$ of degree 3 or more, $\int \sqrt{p(t)}\,dt$ and $\int t\sqrt{p(t)}\,dt$ require the use of "elliptic functions", which I mentioned in class.

9. Describe in Cartesian coordinates the curve which is given in polar coordinates as the solution set to

$$r = 2\cos\theta + 6\sin\theta$$

Answer: $r = 2\cos\theta + 6\sin\theta$

$$\begin{aligned}
&\Rightarrow r^2 = 2r\cos\theta + 6r\sin\theta \\
&\Rightarrow x^2 + y^2 = 2x + 6y \\
&\Rightarrow (x-1)^2 - 1 \;+\; (y-3)^2 - 9 \;=\; 0 \\
&\Rightarrow (x-1)^2 + (y-3)^2 = 10
\end{aligned}$$

So it's the circle of radius $\sqrt{10}$ at (1,3).

10a. Sketch the curve given in polar coordinates by the equation $r = 1 - \cos\theta$. (This curve is called a *cardioid*.)

Answer: [Sketch deleted] The given equation gives a parameterization (by $\theta$) in polar coordinates; the corresponding parameterization in Cartesian coordinates is

$$x = (1-\cos\theta)\cos\theta, \qquad y = (1-\cos\theta)\sin\theta$$

10b. Compute the slope of the line that is tangent to this curve at the point $(r,\theta) = (1, \pi/2)$.

Answer: $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\cos\theta - (cos^2\theta - \sin^2\theta)}{-\sin\theta + (2\cos\theta\sin\theta)}$. When $\theta = \pi/2$ this is $\frac{0-(0-1)}{-1+(2\cdot 0\cdot 1)} = -1$.

11. Is the triangle $PQR$ equilateral if its vertices are at points

$$P = (2,3,5) \qquad Q = (1,5,5) \qquad R = (2,4,7) \quad ?$$

Answer: It's equilateral iff the sides $PQ, QR, RP$ have equal lengths, so use the distance formula:

$$PQ = \mathrm{dist}(P,Q) = \sqrt{(2-1)^2 + (3-5)^2 + (5-5)^2} = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}.$$

Likewise $PR = \sqrt{5}$ while $QR = \sqrt{6}$, so it's *not* equilateral. (But it is isosceles!)

12. Describe the surface in space defined by the equation $x^2 + z^2 - 2x + 4z = 0$

Answer: After completing the squares we see the equation is equivalent to

$$(x-1)^2 + (z-2)^2 = 5.$$

Such points in the $xz$-plane (where $y = 0$) all lie on the circle of radius $\sqrt{5}$ at $(x,z) = (1,-2)$; since $y$ is not constrained by our equation, the solution set is then the set of lines passing perpendicularly through this circle, i.e. the cylinder of radius $\sqrt{5}$ around the line $x = 1, z = -2$.